mathdb:2611.00120v1 17 Nov 2026

DIMENSION- AND LABEL-UNIFORM STABILITY OF GAUSSIAN PROPELLERS

MYEONGJIN KANG

Abstract. Assume the sharp finite Gaussian propeller inequality and its finite equality classification. We prove that the deficit in the squared first-moment functional controls the square of the Gaussian measure distance to a propeller, with one positive constant independent of the ambient dimension and the number of cells. Cells may have arbitrarily small measure, and no masses are prescribed. The proof passes to countable Gaussian score arrays, excludes infinite-support equality cases, and bounds the gain from all additional scores by the three-halves power of their total squared norm and the squared common mean of the leading triple. A quadratic estimate for that triple and a Gaussian margin bound then give the partition estimate. The square-root dependence of distance on deficit is sharp already for three planar sectors. The universal constant is positive but implicit.

1. Introduction

Let γ_d denote standard Gaussian measure on $\mathbb{R}^d$. For a measurable partition $\mathcal{A} = (A_1, \dots, A_k)$, modulo null sets, put

$$z_i = \int_{A_i} x \, d\gamma_d(x), \qquad F(\mathcal{A}) = \sum_i \|z_i\|^2, \qquad B = \frac{3}{2\sqrt{2\pi}}.$$

For $d \geq 2$ and $k \geq 3$, let $\mathcal{P}_{d,k}$ be the partitions formed by three sectors of angle $2\pi/3$ in a two-dimensional linear subspace, extended along its orthogonal complement, with $k-3$ empty cells. All orthogonal transformations and label permutations are allowed. Define

$$d_P(\mathcal{A}) = \inf_{\mathcal{P} \in \mathcal{P}_{d,k}} \frac{1}{2} \sum_{i=1}^{k} \gamma_d(A_i \triangle P_i), \qquad \Delta(\mathcal{A}) = B^2 - F(\mathcal{A}). \tag{1.1}$$

For a fixed alignment, the sum in (1.1) is the probability of a label disagreement.

Our external input is stated explicitly.

Assumption 1.1 (Finite propeller inequality and rigidity)**.** For all finite positive integers d, k and all measurable Gaussian partitions, $F(\mathcal{A}) \leq B^2$. If equality holds, then $d \geq 2$, $k \geq 3$ and $\mathcal{A} \in \mathcal{P}_{d,k}$ modulo null sets.

The inequality is Theorem 1.1 of the pinned preprint [4]. Its scalar and determinant elimination, read at the equality endpoint, together with the three-dimensional uniqueness statement of Heilman, Jagannath and Naor [2, author version, Remark 1.4], supplies the rigidity input. Appendix A records this dependence and the endpoint conditions. We retain Assumption 1.1 in the statement rather than presenting the sharp all-dimensional inequality as a result proved here.

Theorem 1.2. *Under Assumption 1.1, there is a numerical constant $c_P > 0$ such that, for every finite $d \geq 2$, $k \geq 3$ and every measurable partition of $(\mathbb{R}^d, \gamma_d)$,*

$$\Delta(\mathcal{A}) \geq c_P d_P(\mathcal{A})^2. \tag{1.2}$$

The exponent $1/2$ in the corresponding bound on d_P cannot be increased, even for $d = 2$ and $k = 3$.

There are no prescribed masses, lower bounds on nonzero masses or regularity assumptions on the cells. The origin is fixed; translations are not among the symmetries. In particular, after relabelling, the total mass outside three cells is at most $c_P^{-1/2}\sqrt{\Delta}$, and each of the three masses differs from $1/3$ by at most this amount. The infimum over propellers is attained: the choices of labels are finite, orthonormal two-frames form a compact set, and sector indicators are continuous in Gaussian measure under changes of frame.

Khot and Naor [3] introduced the first-moment problem in kernel clustering and developed its Gaussian-score duality. Heilman, Jagannath and Naor [2] proved the sharp inequality in dimension three. A direct quantitative predecessor is Heilman [1, Lemma 2.8]. For three cells, his estimate gives $d_P \le 18\Delta^{1/4}$ after translating his squared L^2 indicator distance into disagreement probability. The present assertion allows an arbitrary finite number of cells, with a constant independent of both that number and the dimension, and gives the sharp exponent.

This uniformity requires control of cells whose masses vanish. For example, let t_1, t_2, t_3 be a centered equilateral triple with $\sum_i \|t_i\|^2 = 1$, and append N mutually orthogonal scores $\sqrt{E/N}\, e_j$ perpendicular to its plane. Multiply the triple by $\sqrt{1-E}$. Every one of the $N+3$ scores wins on a set of positive measure, whereas its expected maximum is at least $B\sqrt{1-E}$. The resulting score partition has deficit at most B^2E, by dual pairing. Thus near-optimality cannot give a uniform bound on the number of nonempty cells.

We prove uniform entry into a neighborhood of the equilateral triple by compactifying sorted score Gram matrices. An infinite-support equality case is excluded by a two-competitor Gaussian integral. Within that neighborhood, the combined gain from the additional scores is cubic in their size. This gives a quadratic score estimate, which transfers to (1.2) through a squared-distance pairing.

2. Countable Gaussian scores

Let $\mathcal{H}$ be a separable real Hilbert space with an isonormal Gaussian process $X(h)$, so that $\mathbb{E}X(h)X(h') = \langle h, h'\rangle$. For $v = (v_i)_{i\ge1}$ with $\sum_i \|v_i\|^2 < \infty$, define

$$M(v) = \mathbb{E}\sup(0, X(v_1), X(v_2), \dots).$$

Finite lists are padded with zero vectors. There is no assertion of an identity-covariance Gaussian random vector taking values in an infinite-dimensional Hilbert space.

Lemma 2.1. *Under Assumption 1.1,*

$$M(v) \le B\Big(\sum_i \|v_i\|^2\Big)^{1/2}. \tag{2.1}$$

For any countable measurable partition of the Gaussian probability space, its first-chaos moments $z_i \in \mathcal{H}$ satisfy $\sum_i \|z_i\|^2 \le B^2$.

Proof. For a finite list in a finite-dimensional span, partition the space according to the largest score, including zero. Pairing the scores with the cell moments and applying Cauchy–Schwarz proves (2.1). For a countable list, $\mathbb{E}\sum_i X(v_i)^2 = \sum_i \|v_i\|^2 < \infty$. Thus the scores lie in ℓ^2 almost surely. Positive maxima of finite heads increase to the supremum, and monotone convergence gives the same bound.

For an event A, Riesz representation defines $z(A)$ by $\langle z(A), h\rangle = \mathbb{E}[\mathbf{1}_A X(h)]$. The map from the indicator to its moment is an L^2 contraction. For a countable partition and any finitely supported vector array (h_i), its assigned score, with zero on untested labels, is at most the positive maximum. Hence $\sum_i \langle h_i, z_i\rangle \le B(\sum_i \|h_i\|^2)^{1/2}$. Duality on finite arrays proves the claimed squared-norm bound. □

mathdb:2611.00120v1 17 Nov 2026

mathdb:2611.00120v1 17 Nov 2026

Lemma 2.2. *Suppose $\sum_i \|v_i\|^2 = 1$ and $M(v) = B$. Its nonzero scores define an almost surely unique winning partition with moments $z_i = Bv_i$. Moreover, $\sum_i z_i = 0$ strongly and $\langle z_i, z_j\rangle \le 0$ for $i \ne j$.*

Proof. The score sequence tends to zero almost surely, so a positive supremum is attained. Choose the least maximizing index. Let A_0 be the event that all scores are nonpositive, and assign vector zero to this label. The pairing is integrable by the $L^2(\ell^2)$ bound. With $z_i = z(A_i)$, Lemma 2.1 gives

$$B = \sum_{i\ge 0}\langle v_i, z_i\rangle \le \Big(\sum_{i\ge 0}\|z_i\|^2\Big)^{1/2} \le B.$$

Equality gives $z_i = Bv_i$, including $z_0 = 0$. If A_0 had positive probability, any fixed nonzero score would be strictly negative almost everywhere on it, contrary to its zero first moment there. Duplicate nonzero scores would leave a later winning cell empty, again contradicting $z_i = Bv_i$. Distinct scores tie only on null Gaussian hyperplanes, and there are countably many such pairs.

The indicators of increasing finite unions of winning cells converge to one in L^2. Contraction of the moment map therefore proves strong convergence of $\sum_i z_i$ to zero. Merging two cells raises the squared moment sum by $2\langle z_i, z_j\rangle$. The bound in Lemma 2.1 forces this inner product to be nonpositive. □

Proposition 2.3. *Every norm-one countable list attaining $M(v) = B$ consists of a centered equilateral triple of vector norms $1/\sqrt{3}$ and zero vectors, up to isometry and relabelling.*

Proof. We first exclude infinite support. Choose three fixed nonzero moments z_1, z_2, z_3 from Lemma 2.2. Put $r_j = \|z_j\|/B$ and $c = \min_{j\le 3} r_j^2 > 0$. If there are infinitely many nonzero moments, choose other indices i with $t = \|z_i\|/B \to 0$. For such an index set

$$e_i = z_i/\|z_i\|, \quad T = X(e_i), \quad a_j = -\frac{\langle z_i, z_j\rangle}{\|z_i\|^2}, \quad w_j = z_j/B + a_j t e_i, \quad Y_j = X(w_j) \quad (j \ne i).$$

The inner-product signs and strong sum give $a_j \ge 0$ and $\sum_{j\ne i} a_j = 1$. The residual variables Y_j are independent of T, with covariance entries

$$H_{jk} = \langle z_j, z_k\rangle/B^2 - a_j a_k t^2 \le 0 \quad (j \ne k), \qquad h_j = H_{jj} = r_j^2 - a_j^2 t^2.$$

On the winning cell A_i, we have $T \ge 0$ and $Y_j \le (1 + a_j)tT$. Its moments satisfy $\mathbb{E}\mathbf{1}_{A_i} T = Bt$ and $\mathbb{E}\mathbf{1}_{A_i} Y_j = 0$.

For distinct competitors j, k, put $D_{jk} = \sqrt{h_j h_k - H_{jk}^2}$. If it is positive, the variables $U = Y_j/((1 + a_j)t)$ and $V = Y_k/((1 + a_k)t)$ have joint density at most $t^2(1 + a_j)(1 + a_k)/(2\pi D_{jk})$. They are independent of T. Taking a positive part and enlarging the integration region gives

$$Bt = \mathbb{E}\mathbf{1}_{A_i}(T + U + V) \le \mathbb{E}\big[\mathbf{1}_{\{T\ge 0,\ U\le T,\ V\le T\}}(T + U + V)_+\big].$$

For $T = s \ge 0$, the integral of $(s + u + v)_+$ on $u, v \le s$ is $(3s)^3/6$. Since $\mathbb{E}T_+^3 = 2/\sqrt{2\pi}$, substitution yields

$$D_{jk} \le \frac{3}{\pi} t(1 + a_j)(1 + a_k). \tag{2.2}$$

If $D_{jk} = 0$, this inequality is automatic. This is the two-competitor integral in [4, Section 4]; the preceding argument verifies the conditions needed for countable lists.

For small t, $h_j \ge c - t^2 > 0$ for $j = 1, 2, 3$. Their correlation matrix is positive semidefinite with nonpositive off-diagonal entries ρ_{jk}. Testing it on $(1, 1, 1)$ gives $\sum_{j<k} |\rho_{jk}| \le 3/2$, and hence $\sum_{j<k} \rho_{jk}^2 \le 3/2$. It follows that

$$\frac{3}{2}(c - t^2)^2 \le \sum_{j<k\le 3} D_{jk}^2 \le 10(3/\pi)^2 t^2. \tag{2.3}$$

For the upper bound, $a_1+a_2+a_3 \le 1$ implies $\sum_{j<k}(1+a_j)^2(1+a_k)^2 < 10$. One direct verification is to increase the sum to one and put $q = a_1a_2 + a_1a_3 + a_2a_3$, $r = a_1a_2a_3$. The expression becomes $9 + 2q + q^2 - 8r \le 9 + 2/3 + 1/9 = 88/9 < 10$. Letting $t \to 0$ in (2.3) is impossible.

Thus the list has finite support. Its winning cells depend only on the finite-dimensional span of those vectors, and their moments attain B^2. Assumption 1.1 makes this partition a propeller. The identities $z_i = Bv_i$ then give precisely the stated equilateral triple and zero vectors. □

3. A uniform neighborhood of the equality arrays

For a finite norm-one list, sort its vectors by decreasing norm and pad with zeros. Write

$$e(v) = B - M(v), \quad \mu = \tfrac{1}{3}(v_1+v_2+v_3), \quad u_i = v_i - \mu \ (i \le 3),$$

$$\tau^2 = \sum_{i>3} \|v_i\|^2, \qquad E = \tau^2 + 3\|\mu\|^2, \qquad S^2 = \sum_{i\le 3} \|u_i\|^2 = 1 - E.$$

Let $\alpha \ge \beta \ge 0$ be the two singular values of the matrix with columns u_1, u_2, u_3, counting zeros when its rank is smaller than two. Define

$$E_0 = (B/1600)^2, \qquad \eta = \inf\{e(v) : E \ge E_0 \text{ or } \beta \le 1/(2\sqrt{2})\}. \tag{3.1}$$

The infimum ranges over all finite dimensions and all finite norm-one lists. Ties in the ordering can be resolved arbitrarily.

Lemma 3.1. *Under Assumption 1.1, $\eta > 0$.*

Proof. Consider a sequence of sorted norm-one lists with $e(v^{(n)}) \to 0$. Their Gram matrices have diagonal entries at most $1/i$. A diagonal subsequence converges entrywise to a positive semidefinite matrix Q with $\operatorname{tr} Q \le 1$. Realize Q as the Gram matrix of a countable list in a separable Hilbert space.

Expected maxima converge along this subsequence. To check the tails, for any $a > 0$ the Gaussian tail bound and integration give

$$\begin{aligned} \mathbb{E} \sup_{i>N} X(v_i^{(n)})_+ &\le a + \sum_{i>N} \frac{e^{-ia^2/2}}{ia} \\ &\le a + \frac{e^{-(N+1)a^2/2}}{(N+1)a(1-e^{-a^2/2})}. \end{aligned} \tag{3.2}$$

The right side tends to zero with $a = \sqrt{4\log(N+1)/(N+1)}$, uniformly in n and in the correlations. The limit list satisfies the same bound. For each finite head, couple the Gaussian vectors by the continuous positive square roots of their covariance matrices. This proves L^1 convergence of the positive maxima, including singular limits. Together with (3.2), it proves convergence of M.

The limit therefore has $M = B$. Lemma 2.1 forces $\operatorname{tr} Q = 1$, and Proposition 2.3 identifies it as the Gram matrix of an equilateral triple and zeros. The decreasing order places that triple first. For the finite lists, E has the head formula $1 - \sum_{i\le 3} \|u_i\|^2$, so it is continuous in those Gram entries, as are the singular values. Thus $E \to 0$ and $\beta \to 1/\sqrt{2}$. Such a sequence cannot remain in the closed set defining (3.1). This proves the lemma. □

The lemma gives a single gap over all dimensions and list lengths. It gives no power law by itself. The rate comes from the local estimate in the next section.

mathdb:2611.00120v1 17 Nov 2026

mathdb:2611.00120v1 17 Nov 2026

4. Local stability with extra scores

Let $\mathcal{T}_{d,k}$ be the norm-one arrays in $(\mathbb{R}^d)^k$ consisting of a centered equilateral triple and zeros, with arbitrary labels. We prove a uniform squared Euclidean distance estimate for the scores.

Lemma 4.1. *If $E < E_0$ and $\beta > 1/(2\sqrt{2})$, then*

$$M(v) \le M_3(u) + 400E^{3/2}, \qquad M_3(u) = \mathbb{E}\max_{i\le 3} X(u_i). \tag{4.1}$$

Proof. The centered triple spans a plane L. A reference equilateral triple of total squared norm one has singular values $1/\sqrt{2}$ and inradius $1/(2\sqrt{3})$. Applying the linear map carrying it to (u_i) gives, for $R = \|g_L\|$,

$$\max_{i\le 3} X(u_i) \ge hR, \qquad h = 1/(4\sqrt{3}).$$

The planar projection of $v_j - \mu$ for $j > 3$ has norm at most $\tau + \|\mu\| \le 2\sqrt{E} \le h/2$. This also bounds the projection of the appended zero competitor minus μ. Write

$$Z = \Big(\sum_{j>3} X(v_j^\perp)^2\Big)^{1/2} + |X(\mu^\perp)|.$$

It is independent of the planar radius R. Comparing every extra score, including zero, with the leading three gives pointwise

$$0 \le \max(0, X(v_1), \dots) - \max_{i\le 3} X(v_i) \le (Z - hR/2)_+.$$

For any centered Gaussian vector with covariance trace q^2, $\mathbb{E}\|G\|^4 \le 3q^4$, hence $\mathbb{E}\|G\|^3 \le 3^{3/4}q^3$. Using $(a+b)^3 \le 4(a^3+b^3)$ and $\mathbb{E}|X(\mu^\perp)|^3 \le 2\sqrt{2/\pi}\|\mu\|^3$ yields

$$\mathbb{E}Z^3 \le 12E^{3/2}.$$

This depends only on the covariance trace, not its rank. For $z \ge 0$, the density $re^{-r^2/2}$ of R gives

$$\mathbb{E}_R(z - hR/2)_+ \le \int_0^{2z/h} (z - hr/2) r\, dr = 32z^3.$$

The expected common score $X(\mu)$ is zero. Integration proves $M(v) - M_3(u) \le 384E^{3/2} \le 400E^{3/2}$. □

Lemma 4.2. *In the same region,*

$$e(v) \ge \frac{B}{4}E + \frac{3}{64\sqrt{2\pi}}(\alpha - \beta)^2. \tag{4.2}$$

There is $t \in \mathcal{T}_{d,k}$ supported on the first three labels with

$$\|v - t\|^2 = 2 - \sqrt{2}(\alpha + \beta) \le 2E + 2(\alpha - \beta)^2. \tag{4.3}$$

Proof. Let ℓ_1, ℓ_2, ℓ_3 be the side lengths of the centered triangle. Integrating its support function over directions gives its perimeter, so polar Gaussian integration gives

$$M_3(u) = \frac{\ell_1 + \ell_2 + \ell_3}{2\sqrt{2\pi}}.$$

The support-function identity follows by integrating each supporting vertex on its normal-angle interval; the endpoint terms sum to the edge lengths. The radial factor is $\mathbb{E}R/(2\pi) = 1/(2\sqrt{2\pi})$. We have $\sum_i \ell_i^2 = 3S^2$ and $\ell_i \le \sqrt{2}S$. Writing the three squared lengths as $S^2 + (\alpha^2 - \beta^2)\cos(2\theta_i - \phi)$, with edge angles separated by $\pi/3$ modulo sign, gives

$$\sum_{i<j} (\ell_i^2 - \ell_j^2)^2 = \frac{9}{2}(\alpha^2 - \beta^2)^2.$$

Consequently,

$$3S - \sum_i \ell_i = \frac{\sum_{i<j}(\ell_i - \ell_j)^2}{3S + \sum_i \ell_i} \geq \frac{3(\alpha^2 - \beta^2)^2}{32S^3},$$
$$BS - M_3(u) \geq \frac{3}{64\sqrt{2\pi}}(\alpha - \beta)^2.$$

For the last line use $(\alpha + \beta)^2 \geq S^2$ and $S \leq 1$. Lemma 4.1, $1 - \sqrt{1 - E} \geq E/2$, and $400\sqrt{E} \leq B/4$ now prove (4.2).

Polar decomposition of the centered three-column matrix aligns it with an equilateral triple whose two singular values are $1/\sqrt{2}$. The common mean is orthogonal, in array space, to this centered triple. Thus the optimal aligned inner product is $(\alpha + \beta)/\sqrt{2}$, giving the equality in (4.3). Put $q = (\alpha - \beta)^2/S^2 \in [0, 1]$ and note that $\sqrt{2}(\alpha + \beta) = 2S\sqrt{1 - q/2}$. The inequalities $1 - S \leq E$, $1 - \sqrt{1 - a} \leq a$, and $S \geq 1/2$ imply

$$2 - \sqrt{2}(\alpha + \beta) \leq 2E + (\alpha - \beta)^2/S \leq 2E + 2(\alpha - \beta)^2.$$

□

Proposition 4.3. *Put*

$$c_0 = \frac{3}{128\sqrt{2\pi}}, \qquad c_V = \min\{c_0, \eta/4\} > 0. \tag{4.4}$$

For every finite norm-one array with $d \geq 2$ and $k \geq 3$,

$$B - M(v) \geq c_V \operatorname{dist}(v, \mathcal{T}_{d,k})^2. \tag{4.5}$$

Proof. In the local region this follows from (4.2)–(4.3). Outside it, Lemma 3.1 gives $e(v) \geq \eta$, whereas the squared distance between two unit arrays is at most four. Sorting merely permutes labels and does not change the distance. □

5. From scores to partitions

Proof of Theorem 1.2. Suppose first that $F = F(\mathcal{A}) > 0$, and put $v_i = z_i/\sqrt{F}$. Assigning each point its actual label shows $M(v) \geq \sqrt{F}$. Choose a closest $t \in \mathcal{T}_{d,k}$ in Proposition 4.3; the minimum exists by compactness. Its winning cells define a propeller $\mathcal{P}$ with the corresponding labels. The mean loss from assigning the label of $\mathcal{A}$ to these scores is

$$\begin{aligned} J &= \mathbb{E}[\max_i X(t_i) - X(t_{\mathcal{A}(g)})] = B - \sum_i \langle t_i, z_i \rangle \\ &= B - \sqrt{F} + \frac{\sqrt{F}}{2}\|v - t\|^2 \\ &\leq \left(1 + \frac{B}{2c_V}\right)(B - \sqrt{F}) \leq \left(B^{-1} + (2c_V)^{-1}\right)\Delta. \end{aligned} \tag{5.1}$$

The equality uses that both arrays have norm one.

Let $G(g)$ be the difference between the winning score of t and the highest score with a different label. The three active pair differences are standard normal. For $0 < s \leq 1$, their three absolute slabs have total probability at most $3\sqrt{2/\pi}s$. All inactive scores are the same zero score. An inactive score is within s of the winner only if $R/(2\sqrt{3}) \leq s$, where R is the planar Gaussian radius. That event has probability at most $6s^2$. Therefore

$$\mathbb{P}(G \leq s) \leq 3\sqrt{2/\pi}s + 6s^2 \leq 9s. \tag{5.2}$$

mathdb:2611.00120v1 17 Nov 2026

For $k = 3$ the zero-score term is unnecessary, but harmless. If ϵ is the probability of a label disagreement, then $J \geq s(\epsilon - 9s)$. Take $s = \epsilon/18$ to obtain $J \geq \epsilon^2/36 \geq d_P(\mathcal{A})^2/36$. Together with (5.1), this proves (1.2) with

$$c_P = \left[36\left(B^{-1} + (2c_V)^{-1}\right)\right]^{-1}. \tag{5.3}$$

If $F = 0$, then $\Delta = B^2$ and $d_P \leq 1$, and the same constant works because $c_P \leq B/36 < B^2$.

For sharpness, in the plane take consecutive sectors of openings $2\pi/3 + t$, $2\pi/3 - t$ and $2\pi/3$, for small $t > 0$. A sector of angle a has first-moment norm $\sin(a/2)/\sqrt{2\pi}$. It follows that

$$\Delta = \frac{1 - \cos t}{4\pi}, \qquad d_P = \frac{t}{2\pi}. \tag{5.4}$$

For the distance, the largest cell has excess mass $t/(2\pi)$ over $1/3$, a lower bound for disagreement under every alignment. Keeping two original boundaries fixed attains that lower bound. Thus $\Delta/d_P^2 \to \pi/2$, excluding any uniform estimate $d_P \leq C\Delta^a$ with $a > 1/2$. □

The constant (5.3) is specified through the positive infimum (3.1); no decimal value or optimal constant is asserted. All list lengths in Theorem 1.2 are finite. Countable arrays enter only to prove the uniform finite-list estimate.

Appendix A. The finite equality input

We explain how the finite rigidity clause in Assumption 1.1 relates to the cited sources. This keeps the quantitative proof separate from the underlying sharp theorem. The revision of [4] is fixed in the bibliography. Its main theorem gives the finite bound B^2, and its `ext:duality` gives the equivalent finite Gaussian-score bound. The equality case is obtained from its proof as follows.

For a finite equality partition, dual pairing shows that its cells are winning cells for their moment scores. A cell with zero moment cannot have positive measure: on such a winning cell every nonzero score is nonpositive, whereas the moment of that cell is zero. Merging any pair of cells shows that the off-diagonal moment products are nonpositive. Merge zero-product pairs until every product is strictly negative. The Gram matrix then satisfies

$$a^T Q a = \sum_{i<j} (-\langle z_i, z_j \rangle)(a_i - a_j)^2,$$

so its rank is $m - 1$, where m is the remaining active count.

For $m \geq 5$, the scalar estimates in [4], Lemma 3.1, and the two residual inequalities in its Corollary 4.2 remain valid at objective value $C = B^2$. In their notation,

$$r_i = \|z_i\|/B, \quad P_i = \gamma(A_i), \quad r_1 \geq \cdots \geq r_m = t > 0, \quad L = r_1^2 + r_2^2, \quad c = r_3^2,$$

they assert

$$r_i \leq 2/3, \qquad P_i \geq A r_i^2, \qquad P_i \geq 0.415 r_i + 0.15 r_i^2,$$
$$A = 0.929 \ (m = 5), \qquad A = 0.884 \ (m \geq 6),$$
$$(c - t^2)(L - c - 2t^2) \leq 8(3/\pi)^2 t^2, \qquad \tfrac{3}{2}(c - t^2)^2 \leq 10(3/\pi)^2 t^2.$$

Here all terminating decimals are exact rational numbers. The halfspace estimate $r_i < 2/3$ under strict excess becomes $r_i \leq 2/3$; the scalar intervals already include this endpoint. The one-score deletion and translation inequalities use optimality and the full-rank score structure. Replacing $C > B^2$ by $C = B^2$ preserves their non-strict lower bounds. The pairwise density integral likewise preserves its non-strict conclusion, as in (2.2) above.

The elimination in [4, Section 5 and Appendix A] uses only the displayed inequalities and the normalizations $\sum_i r_i^2 = \sum_i P_i = 1$. It obtains $t < 0.12$ and $c > 2.6t$ with strict rational margins. These contradict

$$10(3/\pi)^2 t^2 \geq 1.5(c - t^2)^2 > 1.5(2.6 - 0.12)^2 t^2,$$

mathdb:2611.00120v1 17 Nov 2026

since $1.5(2.6-0.12)^2-10(3/\pi)^2>0.1066$. Thus its exclusion still applies at equality. We use those scalar estimates and their finite certificates as part of the pinned source input, not as numerical experiments of this paper.

For $m\le 4$, the score partition factors through a space of dimension at most three. The uniqueness statement in the Princeton author version of [2, Remark 1.4] makes it a three-cell propeller. This remark is version-specific and is absent from arXiv v2. No earlier zero-product merge was possible: the last such merge would start with a four-active-cell equality partition, which also factors through dimension at most three and contradicts that uniqueness statement. This gives finite rigidity before merging. The computer-assisted three-dimensional theorem is an external input; its computation is not rerun here.

Acknowledgements

The author used OpenAI language models for proof development, literature searches, computational checks, and manuscript preparation.

References

[1] S. Heilman, Euclidean partitions optimizing noise stability, *Electron. J. Probab.* **19** (2014), no. 71, 1–37. Author version, Lemma 2.8.

[2] S. Heilman, A. Jagannath and A. Naor, Solution of the propeller conjecture in $\mathbb{R}^3$, *Discrete Comput. Geom.* **50** (2013), 263–305. doi:10.1007/s00454-013-9530-0. The uniqueness input is Remark 1.4 in the Princeton author version.

[3] S. Khot and A. Naor, Approximate kernel clustering, *Mathematika* **55** (2009), 129–165. Publisher version.

[4] OpenAI, *The Gaussian propeller bound in every dimension*, September 24, 2026, preprint in the `openai/math` repository, revision `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`, pinned manuscript.

Independent researcher

mathdb:2611.00120v1 17 Nov 2026