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WEIGHTED CAPACITY FOR FOUR-STATE RECONSTRUCTION WITH VARIABLE EDGE CHANNELS

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Abstract. Assuming a specified moment-saving inequality for a preserved class of four-state posterior laws, we characterize reconstruction on deterministic locally finite trees with variable symmetric ferromagnetic edge channels by positivity of a weighted nonlinear capacity. The channel parameters may approach either zero or one, and perfectly transmitting edges are allowed. We give quantitative comparisons between capacity and limiting posterior information with constants independent of the tree, its degrees, and the edge parameters. The proof uses local posterior estimates valid without channel smoothing and a flow estimate controlled by branch splitting. The capacity is unchanged by replacing a chain by its composite channel; on a ray, it gives the exact criterion that the product of the channel parameters is positive.

1. Introduction

Reconstruction on a tree asks whether spins observed far from the root retain information about its spin. For irregular trees, nonlinear capacities can distinguish models with the same exponential growth rate. Pemantle and Peres [4] developed capacity criteria and concave recursions for critical Ising models on trees. The dependence of reconstruction on the number of states is studied, in particular, by Sly [5] and Mossel, Sly, and Sohn [1].

For the symmetric four-state model, the fixed-channel capacity theorem of [2] uses a moment-saving inequality for a preserved class of posterior laws from [3]. We take this inequality and its closure properties as an explicit assumption. Starting from that input, we prove a capacity criterion for variable edge channels on every locally finite tree. The main issue is uniformity as channel parameters approach one: estimates obtained by bounding posterior denominators away from zero no longer suffice. A bound using the posterior numerator instead removes this restriction. A second estimate uses flow splitting to control the accumulated information loss, including along long chains with nearly perfect transmission.

1.1. The model and capacity.

Let T be a deterministic rooted locally finite tree with root ρ. Every vertex has finitely many children, but their numbers need not have a common bound. Write $L_n = \{v : |v| = n\}$ and $\mathrm{ch}(v)$ for the children of v. The tree may be finite and may have dead ends. Give the root a uniform spin in $[4] = \{1, 2, 3, 4\}$. Each edge $e = (v, w)$ transmits its parent spin through the channel

$$P_e(j \mid i) = \lambda_e \mathbf{1}_{\{i=j\}} + \frac{1 - \lambda_e}{4}, \qquad 0 < \lambda_e \le 1. \tag{1.1}$$

The edge randomness is independent, and the tree and all parameters are known to the observer. Set $\mathsf{u} = (1/4, 1/4, 1/4, 1/4)$ and

$$\begin{aligned} p^{(n)} &= \mathbb{P}(\sigma_\rho \in \cdot \mid \sigma_{L_n}), \\ a_n &= \mathbb{E}\,\mathrm{TV}(p^{(n)}, \mathsf{u}), \qquad m_n = 4\mathbb{E}\|p^{(n)} - \mathsf{u}\|_2^2. \end{aligned} \tag{1.2}$$

An empty observation has posterior u. Both sequences decrease by data processing, so their limits a_∞, m_∞ exist. Reconstruction means $a_\infty > 0$.

Put

$$r_e = \lambda_e^2, \qquad R_\rho = 1, \qquad R_w = \prod_{e \in [\rho, w]} r_e.$$

A flow θ is a nonnegative assignment to edges, conserved at each nonroot vertex, with finite mass $|\theta| = \sum_{w \in \mathrm{ch}(\rho)} \theta(\rho, w)$. Conservation forces zero incoming flow at a leaf. Let ∂T be the infinite rays from ρ, and define

$$V(\theta) = \sup_{\xi \in \partial T} \sum_{e=(v,w) \in \xi} (1 - r_e^2) \left(\frac{\theta(e)}{R_w} \right)^2, \tag{1.3}$$

$$C(T) = \sup\{|\theta| : V(\theta) \le 1\}. \tag{1.4}$$

The supremum over no rays in (1.3) is zero. The zero flow is allowed, and $C(T)$ may be infinite. This is an L^3 nonlinear capacity in the convention where the path potential has exponent $p-1$; the corresponding edge resistances are $\sqrt{1 - r_e^2}/R_w$. When all channels have one parameter $0 < \lambda < 1$, its positivity agrees with the capacity in [2].

Theorem 1.1. *Assume the posterior-law inequality and closure properties in Assumption 2.1. For the model* (1.1) *on every deterministic rooted locally finite tree,*

$$a_\infty > 0 \quad \Longleftrightarrow \quad C(T) > 0.$$

More precisely, put

$$K = \frac{1}{3000}, \qquad \kappa = \frac{2K}{(1 + 9K)^3}.$$

Then

$$\frac{1}{20000} \frac{C(T)}{\sqrt{1 + C(T)^2}} \le m_\infty \le \min\left\{3, \frac{C(T)}{\sqrt{\kappa}}\right\}. \tag{1.5}$$

The fraction on the left is interpreted as one if $C(T) = \infty$. All constants are independent of the degrees and channel parameters.

The equivalence follows from these bounds and

$$\frac{m_n}{8} \le a_n \le \frac{\sqrt{m_n}}{2}. \tag{1.6}$$

Indeed, for $d = p - \mathsf{u}$, $\|d\|_2^2 \le \|d\|_1$ and $\|d\|_1 \le 2\|d\|_2$, and Jensen's inequality gives the upper bound after taking expectations.

The proof of the lower bound is the part that permits parameters arbitrarily close to one. The degree-independent upper bound follows from a cubic change of variable in the binary moment-saving inequality. Section 5 explains the weight in (1.3), treats zero channels, and shows why local finiteness cannot simply be omitted.

2. Posterior laws and uniform local estimates

An experiment has a uniform input $I \in [4]$ and an observation Y. Its posterior law is the distribution of $p = \mathbb{P}(I \in \cdot \mid Y)$ under the marginal law of Y. Call this law symmetric if it is invariant under every permutation of the four coordinates. Define

$$L = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad x = Lp, \qquad A(p) = |x|^2.$$

Since $p = \mathsf{u} + L^{\mathsf{T}} x/4$, we have $A(p) = 4|p - \mathsf{u}|^2$. For a posterior law μ, write

$$m(\mu) = \mathbb{E}_\mu A, \qquad \tau(\mu) = \mathbb{E}_\mu x_1 x_2 x_3, \qquad h(\mu) = |\tau(\mu)|.$$

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Always $|x_i| \le 1$, $0 \le m \le 3$, and $h \le m$. For a symmetric law, coordinate permutations and even sign changes give

$$\mathbb{E}x_i = 0, \qquad \mathbb{E}x_i x_j = \frac{m}{3}\mathbf{1}_{\{i=j\}}. \tag{2.1}$$

Every degree-three coordinate monomial other than $x_1x_2x_3$ has zero expectation. Every monomial of degree at least two has absolute value at most $|x|^2$: retain two factors and use $|x_i| \le 1$.

2.1. **Channel steps and products.** A channel step of parameter $t \in [0,1]$ replaces a posterior p by $\mathsf{u} + t(p - \mathsf{u})$. This follows by composing the experiment with the symmetric channel P_t and applying Bayes' formula under the uniform prior. It sends

$$(x, m, \tau, h) \longmapsto (tx, t^2 m, t^3\tau, t^3 h). \tag{2.2}$$

For two experiments independent conditional on the same input, denote the product posterior law by $\mu \star \nu$. To compute it, first draw $p \sim \mu, q \sim \nu$ independently under their marginal laws. Set

$$Z = 4\sum_{i=1}^{4} p_i q_i, \qquad P_i = \frac{4p_i q_i}{Z} \quad (Z > 0). \tag{2.3}$$

The joint observation marginal has density Z with respect to the product of the two marginals, and $\mathbb{E}Z = 1$. Thus $\mathbb{E}_{\mu\star\nu} f = \mathbb{E}_{\mu\otimes\nu}[Zf(P)]$. Assign any posterior on $Z = 0$, where the weighted contribution is zero. These formulas also show that channel steps and products preserve symmetry.

2.2. **The assumed moment saving.** Here is the complete input used from [3, 2].

Assumption 2.1. There is a class $\mathcal{C}$ of symmetric posterior laws containing the revealing law, uniform on the four point masses, and the uninformative law δ_{u}. It is closed under channel steps with every $t \in [0,1]$ and under products of experiments independent conditional on the same uniform input. For $\mu, \nu \in \mathcal{C}$,

$$m(\mu \star \nu) \le m(\mu) + m(\nu) - \frac{\mathbb{E}_\mu[A^2]m(\nu) + m(\mu)\mathbb{E}_\nu[A^2]}{1000}. \tag{2.4}$$

The class specified in [3, Proposition 3.2] consists of symmetric laws with $\mathbb{E}H(p) \ge 0$, where, with $B = x_1x_2x_3$ and $J = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$,

$$H = B - \frac{A^2}{10} - \frac{3J}{20} + \frac{3AB}{10} - \frac{A^3}{50} + \frac{7AJ}{100} - \frac{51B^2}{100}. \tag{2.5}$$

That proposition asserts the closure properties and (2.4); [2] restates them as its companion input. We use the repository versions fixed in the bibliography and do not prove Assumption 2.1 here.

Every finite subtree experiment in this paper has law in $\mathcal{C}$. At a terminal we observe its spin, possibly after an independent channel step, or observe nothing. Induction toward the root uses (2.2) and conditionally independent products. This also proves membership for every partial product of child experiments. No closure under limits is assumed or used.

2.3. **Estimates without smoothing.** The following estimates require only symmetry.

Lemma 2.2. *Let μ, ν be symmetric four-state posterior laws, and put $a = m(\mu)$, $b = m(\nu)$, $h_\mu = h(\mu)$, $h_\nu = h(\nu)$. Then*

$$m(\mu \star \nu) \ge a + b - 7(h_\mu b + a h_\nu), \tag{2.6}$$

$$h(\mu \star \nu) \le h_\mu + h_\nu + 460ab. \tag{2.7}$$

The bounds include laws supported on the boundary of the simplex.

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Proof. All expectations below are under independent marginal coordinates $x = Lp$, $y = Lq$. Put $s = x \cdot y$ and, for $\{i, j, k\} = \{1, 2, 3\}$, set

$$N_i = x_i + y_i + x_j y_k + x_k y_j.$$

Equation (2.3) gives $Z = 1 + s$, $LP = N/Z$ on $Z > 0$. In particular,

$$0 \le Z \le 4, \qquad |N_i| \le Z, \qquad N = 0 \text{ when } Z = 0. \tag{2.8}$$

The last assertion follows also from the unnormalized Bayes numerator.

Write $N = x + y + W$, where $W_i = x_j y_k + x_k y_j$, and put $\alpha = \tau(\mu)$, $\gamma = \tau(\nu)$. Symmetry and independence give

$$\mathbb{E}|W|^2 = \frac{2}{3}ab, \quad \mathbb{E}s^2 = \frac{1}{3}ab, \quad \mathbb{E}[s\, x \cdot W] = 2\alpha b, \quad \mathbb{E}[s\, y \cdot W] = 2a\gamma, \quad \mathbb{E}[s|W|^2] = 6\alpha\gamma.$$

For example, $x \cdot W = 2(x_1x_2y_3 + x_1x_3y_2 + x_2x_3y_1)$; after multiplication by s, only the third moment with three distinct indices survives in x. Expanding $|N|^2(1-s)$, terms of degree one in either vector vanish, and the terms $\mathbb{E}|W|^2$ and $2\mathbb{E}s^2$ cancel. Consequently

$$m(\mu \star \nu) = a + b - 4(\alpha b + a\gamma) - 6\alpha\gamma + \mathbb{E}\frac{|N|^2 s^2}{1+s}. \tag{2.9}$$

Here we used $(1+s)^{-1} = 1 - s + s^2/(1+s)$. The last term is nonnegative, and extends by zero at $Z = 0$ because $|N|^2/Z \le 3Z$. Since $|\alpha\gamma| \le (h_\mu b + a h_\nu)/2$, this proves (2.6).

For the third moment,

$$\tau(\mu \star \nu) = \mathbb{E}\frac{N_1N_2N_3}{(1+s)^2}, \qquad \frac{1}{(1+s)^2} - (1-2s) = \frac{s^2(3+2s)}{(1+s)^2}.$$

The absolute remainder after multiplication by $N_1N_2N_3$ is at most $36s^2$, by (2.8), since $|N_1N_2N_3| \le Z^3$ and $|3 + 2s| \le 9$. Its expectation is at most $12ab$. All expressions extend by zero at $Z = 0$.

In the polynomial $N_1N_2N_3(1 - 2s)$ the only pure terms are $x_1x_2x_3$ and $y_1y_2y_3$. Mixed monomials of degree one in one vector have zero expectation. Every remaining monomial has absolute expectation at most ab, by the monomial bound above and independence. The sum of the absolute coefficients is at most $4^3(1 + 2 \cdot 3) = 448$. Thus the polynomial expectation differs from $\alpha + \gamma$ by at most $448ab$. Adding the remainder and taking absolute values proves (2.7). □

3. From reconstruction to capacity

Lemma 3.1. *Under Assumption 2.1, the increasing function $g(t) = t + Kt^3$, with $K = 1/3000$, satisfies*

$$g\big(m(\mu_1 \star \cdots \star \mu_d)\big) \le \sum_{j=1}^{d} g(m(\mu_j)) \tag{3.1}$$

for every finite collection of conditionally independent experiments whose laws lie in $\mathcal{C}$.

Proof. For two laws put $a = m(\mu)$, $b = m(\nu)$, and $d_0 = m(\mu \star \nu)$. Jensen's inequality and (2.4) imply

$$0 \le d_0 \le a + b - cab(a+b) \le a + b, \qquad c = 1/1000.$$

Use the first upper bound for the linear term and the second for the cubic term to obtain

$$g(d_0) \le a + b - cab(a+b) + K(a+b)^3 = g(a) + g(b) + (3K - c)ab(a+b) = g(a) + g(b).$$

This step does not require $a + b \le 3$. Closure under products allows iteration. The empty product is uninformative. □

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Let $m_v^{(n)}$ be the information about σ_v from its descendants at distance n, and let $m_v = \lim_n m_v^{(n)}$. The limit exists by data processing and lies in $[0,3]$. Apply Lemma 3.1 at finite depth and pass to the limit through the finitely many children of each vertex. With $y_v = g(m_v)$, we get

$$\begin{aligned} y_v &\le \sum_{w\in \mathrm{ch}(v)} g(r_e m_w) \\ &= \sum_w r_e y_w - K \sum_w r_e (1-r_e^2) m_w^3 \\ &\le \sum_w r_e y_w \left(1 - \frac{\kappa}{2}(1-r_e^2) y_w^2\right) \\ &\le \sum_w \frac{r_e y_w}{\sqrt{1+\kappa(1-r_e^2)y_w^2}}. \end{aligned} \tag{3.2}$$

Here $e=(v,w)$, $y_w \le (1+9K)m_w$, and $(1+2t)^{-1/2} \ge 1-t$ for $t \ge 0$. No limiting posterior law has been assumed to lie in $\mathcal{C}$.

We now construct a flow directly from these labels. For each vertex choose coefficients

$$0 \le d_e \le \frac{r_e}{\sqrt{1+\kappa(1-r_e^2)y_w^2}} \quad \text{such that} \quad y_v = \sum_w d_e y_w.$$

If the upper-bound sum is positive, scale all its coefficients by the ratio of y_v to that sum; otherwise choose them all zero. Put $\ell_\rho = 1$, $\ell_w = \ell_v d_e$, and $\theta(e) = \ell_w y_w$. This is a conserved flow of mass y_ρ. With $z_v = \ell_v / R_v$, the coefficient bound gives

$$\kappa(1-r_e^2)\left(\frac{\theta(e)}{R_w}\right)^2 \le z_v^2 - z_w^2.$$

Summing along any ray gives a bound of $z_\rho^2 = 1$. Thus $\sqrt{\kappa}\,\theta$ is admissible and

$$C(T) \ge \sqrt{\kappa}\, g(m_\infty) \ge \sqrt{\kappa}\, m_\infty. \tag{3.3}$$

This proves the upper bound in Theorem 1.1. The construction also covers zero labels, dead ends, and perfectly transmitting edges.

4. From capacity to reconstruction

Suppose θ is an admissible flow of finite mass $A_0 > 0$. For a parameter $\varepsilon > 0$ to be chosen, set

$$\Theta = \varepsilon\theta, \qquad M = \varepsilon A_0, \qquad Q = 1 + A_0^2.$$

Write $\Theta(v)$ for the incoming flow at $v \ne \rho$, put $\Theta(\rho) = M$, and define $X_v = \Theta(v)/R_v$. Conservation gives

$$X_v = \sum_{w\in \mathrm{ch}(v)} r_e X_w. \tag{4.1}$$

4.1. **A bounded profile and degraded observations.** Every positive-flow vertex extends to an infinite ray, by repeatedly choosing a positive-flow child. Along a positive-flow edge, $r_e X_w \le X_v$, so

$$X_w^2 - X_v^2 \le (1-r_e^2) X_w^2.$$

Telescoping from the root to a vertex, and extending to a ray, yields

$$\sup_{v:\Theta(v)>0} X_v^2 \le M^2 + \varepsilon^2 = \varepsilon^2 Q. \tag{4.2}$$

Choose $\varepsilon\sqrt{Q} \le 1$. Fix $n \ge 1$. Independently pass each observed spin at $v \in L_n$ through a channel with parameter $\sqrt{X_v/3}$, using parameter zero when $X_v = 0$. Let $\widehat{m}_v, \widehat{h}_v$ be the information and

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absolute third moment of the resulting descendant observation about σ_v, for $|v| \le n$. At the terminals,

$$\widehat{m}_v = X_v, \qquad \widehat{h}_v = (X_v/3)^{3/2}. \tag{4.3}$$

All these experiments belong to $\mathcal{C}$. Subadditivity from (2.4) and (4.1) gives inductively $\widehat{m}_v \le X_v$. Zero-flow branches are uninformative and may be omitted. All computations at depth n involve finitely many vertices, because T is locally finite.

At a positive-flow vertex define

$$t_{vw} = \frac{\Theta(w)}{\Theta(v)}, \qquad \beta_v = 1 - \sum_w t_{vw}^2, \qquad U_v = \frac{\widehat{h}_v}{X_v}. \tag{4.4}$$

The sums are over its positive-flow children, and $\sum_w t_{vw} = 1$.

Lemma 4.1. *For every positive-flow vertex v with $|v| < n$, put*

$$D_v = \sum_w r_e \widehat{m}_w - \widehat{m}_v.$$

With the universal constant $C_0 = 2000$,

$$U_v \le \sum_w t_{vw}\lambda_e U_w + C_0 X_v \beta_v, \tag{4.5}$$

$$0 \le \frac{D_v}{X_v} \le C_0 \left[X_v \sum_w t_{vw}(1 - t_{vw})\lambda_e U_w + X_v^2 \beta_v \right]. \tag{4.6}$$

Proof. Order the children arbitrarily and write $b_i = r_e \widehat{m}_w$, $h_i' = \lambda_e^3 \widehat{h}_w$ for their channel-transformed moments. Set $S = \sum_i b_i$ and $P_2 = \sum_{i<j} b_i b_j$. Then

$$b_i \le t_i X_v, \qquad S \le X_v, \qquad P_2 \le \frac{X_v^2 \beta_v}{2}. \tag{4.7}$$

Sequential application of (2.7), using subadditivity for each partial product, gives

$$\widehat{h}_v \le \sum_i h_i' + 460 P_2.$$

Since $h_i' = t_i X_v \lambda_e U_w$, this proves (4.5).

By (2.6), the deficit at each multiplication is at most seven times the sum of the two cross products of a second and an absolute third moment. Summing over the ordered products yields

$$D_v \le 7 \sum_i h_i'(S - b_i) + 7 \cdot 460 \sum_{i<j<k} b_i b_j b_k.$$

The triple sum is at most $X_v P_2$. Also $S - b_i = \sum_{j \ne i} b_j \le X_v(1 - t_i)$. After division by X_v, (4.7) bounds the triple term by $1610 X_v^2 \beta_v$, while the pair term is at most $7 X_v \sum_i t_i(1 - t_i)\lambda_e U_w$. Subadditivity gives $D_v \ge 0$. Both coefficients are bounded by C_0. With one positive-flow child, $\beta_v = D_v = 0$. □

4.2. **Energy from branch splitting.** Choose a terminal $v \in L_n$ with probability $\Theta(v)/M$, and let $V_0 = \rho, V_1, \dots, V_n = v$ be its ancestral path. Denote expectation for this choice by $\mathbf{E}$; it is separate from the expectations defining posterior moments. Let $\mathcal{F}_i = \sigma(V_0, \dots, V_i)$ and write

$$X_i = X_{V_i}, \quad t_{i+1} = t_{V_i V_{i+1}}, \quad \alpha_i = 1 - t_{i+1},$$

with λ_{i+1}, r_{i+1} the parameters of that edge. Conditional on $\mathcal{F}_i$, the next child is chosen with probabilities $t_{V_i w}$, so $\mathbf{E}[\alpha_i \mid \mathcal{F}_i] = \beta_{V_i}$.

Lemma 4.2. *For every n,*

$$\mathbf{E} \sum_{i=0}^{n-1} \alpha_i X_i^2 \le \varepsilon^2 Q. \tag{4.8}$$

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Proof. Using $r_e X_w = t_{vw} X_v$, direct conditional expectation gives

$$X_i^2 - \mathbf{E}[X_{i+1}^2 \mid \mathcal{F}_i] + \mathbf{E}[(1 - r_{i+1}^2) X_{i+1}^2 \mid \mathcal{F}_i] = X_i^2 \left(1 - \sum_w t_{V_i w}^3\right) \geq X_i^2 \beta_{V_i}.$$

Sum over i and take expectations. The first two terms telescope to $M^2 - \mathbf{E}X_n^2$. Every sampled path extends to an infinite ray, so its sum of $(1-r_{i+1}^2)X_{i+1}^2$ is at most ε^2 by admissibility. Dropping the negative terminal term and replacing β_{V_i} by its conditional expectation proves (4.8). □

For $0 \leq i < j \leq n$ put

$$K_{ij} = \prod_{k=i+1}^{j} \sqrt{t_k}, \qquad K_{ii} = 1.$$

On every sampled path,

$$\sum_{j=i+1}^{n-1} \alpha_j K_{ij} \leq 2, \qquad \sum_{i=0}^{j-1} \alpha_i K_{ij} \leq 2 \quad (1 \leq j \leq n). \tag{4.9}$$

Indeed, $1 - t \leq 2(1 - \sqrt{t})$ for $0 \leq t \leq 1$. For the first bound use $\alpha_j K_{ij} \leq 2(K_{ij} - K_{i,j+1})$ and telescope. For the second use $\alpha_i K_{ij} \leq 2(K_{i+1,j} - K_{ij})$ and telescope. In particular, the second bound includes the terminal index $j = n$.

Lemma 4.3. *On every sampled path,*

$$\sum_{0 \leq i < j < n} \alpha_i \alpha_j X_i X_j \prod_{k=i+1}^{j} \lambda_k \leq 2 \sum_{i=0}^{n-1} \alpha_i X_i^2. \tag{4.10}$$

Proof. The profile relation gives $X_j = X_i \prod_{k=i+1}^{j} (t_k/\lambda_k^2)$. Hence

$$X_i X_j \prod_{k=i+1}^{j} \lambda_k = X_i^{3/2} X_j^{1/2} K_{ij} \leq \left(\frac{3}{4} X_i^2 + \frac{1}{4} X_j^2\right) K_{ij}.$$

Multiply by $\alpha_i \alpha_j$ and sum. The row and column bounds in (4.9) bound the two terms by, respectively, $\frac{3}{2} \sum_i \alpha_i X_i^2$ and $\frac{1}{2} \sum_j \alpha_j X_j^2$. □

4.3. **The total information loss.** Unfold (4.5) down to level n and insert it into (4.6). Taking path expectations gives

$$\begin{aligned} \mathbf{E} \sum_{i<n} \frac{D_{V_i}}{X_i} \leq{}& C_0 \varepsilon^2 Q \\ &+ C_0^2 \mathbf{E} \sum_{i<j<n} \alpha_i \alpha_j X_i X_j \prod_{k=i+1}^{j} \lambda_k \\ &+ C_0 \mathbf{E} \sum_{i<n} \alpha_i X_i \left(\prod_{k=i+1}^{n} \lambda_k\right) U_{V_n}. \end{aligned} \tag{4.11}$$

To justify the conditional-expectation step explicitly, (4.6) first contributes $X_i \mathbf{E}[\alpha_i \lambda_{i+1} U_{V_{i+1}} \mid \mathcal{F}_i]$. Iteration of (4.5) from $i+1$ replaces $U_{V_{i+1}}$ by its terminal contribution and the sum of $C_0 X_j \beta_{V_j}$ with the intervening channel products. For $i < j$, its multiplier $\alpha_i X_i X_j \prod_{k=i+1}^{j} \lambda_k$ is $\mathcal{F}_j$-measurable, so β_{V_j} may be replaced by α_j inside expectation. The direct $X_i^2 \beta_{V_i}$ term is bounded by Lemma 4.2. This proves (4.11) using only finite sums.

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By Lemmas 4.2 and 4.3, the middle term in (4.11) is at most $2C_0^2\varepsilon^2 Q$. At the terminal, (4.3) gives $U_{V_n} = \sqrt{X_n}/(3\sqrt{3})$, and therefore

$$X_i \left(\prod_{k=i+1}^{n} \lambda_k \right) U_{V_n} = \frac{X_i^{3/2}}{3\sqrt{3}} K_{in}.$$

Use (4.2) and the terminal case of (4.9) to bound its sum with α_i by $2\varepsilon^{3/2}Q^{3/4}/(3\sqrt{3})$. It follows that

$$\mathbf{E} \sum_{i<n} \frac{D_{V_i}}{X_i} \le (C_0 + 2C_0^2)\varepsilon^2 Q + \frac{2C_0}{3\sqrt{3}} \varepsilon^{3/2} Q^{3/4}. \tag{4.12}$$

Choose $\varepsilon = 10^{-4}/\sqrt{Q}$. Then the right side is

$$(2000 + 2 \cdot 2000^2)10^{-8} + \frac{4000}{3\sqrt{3}} 10^{-6} = 0.08002 + \frac{1}{750\sqrt{3}} < \frac{1}{2}.$$

This choice also satisfies $\varepsilon\sqrt{Q} \le 1$.

Finally, multiply the definition of D_v by R_v and sum over the finite truncation at depth n. The intermediate terms cancel, and (4.3) gives

$$\begin{aligned} \widehat{m}_\rho &= \sum_{|v|=n} R_v X_v - \sum_{|v|<n} R_v D_v \\ &= M - M\mathbf{E} \sum_{i<n} \frac{D_{V_i}}{X_i} \ge \frac{M}{2}. \end{aligned}$$

Zero-flow vertices contribute zero, so the displayed ratios are only taken on sampled paths. Every level has total flow M. The original level observation is more informative than the degraded one. Therefore, for every $n \ge 1$,

$$m_n \ge \frac{\varepsilon A_0}{2} = \frac{1}{20000} \frac{A_0}{\sqrt{1 + A_0^2}}.$$

Pass to the limit in n, then take admissible flow masses tending to $C(T)$, or to infinity when $C(T) = \infty$. This proves the lower bound in (1.5). If $C(T) = 0$, that lower bound is automatic. Together with (3.3) and (1.6), the proof of Theorem 1.1 is complete.

5. Series reduction and boundary cases

5.1. **Chains and rays.** The weight in (1.3) satisfies the exact identity

$$\frac{1 - r_e^2}{R_w^2} = \frac{1}{R_w^2} - \frac{1}{R_v^2}, \qquad e = (v, w). \tag{5.1}$$

Along a finite chain whose internal vertices have one child, a conserved flow has constant value. Its potential contribution therefore telescopes to the contribution of one edge with channel parameter equal to the product of the parameters on that chain. The channels themselves obey $P_s P_t = P_{st}$. Thus replacing such a chain by its composite channel preserves the capacity, as well as the experiment between its endpoints. For a constant parameter $0 < \lambda < 1$, the capacity here equals $(1 - \lambda^4)^{-1/2}$ times the capacity normalized in [2].

On a single infinite ray, put $P_n = \prod_{j=1}^{n} \lambda_j$ and $P_\infty = \lim_n P_n$. The unit flow has potential

$$\lim_{n \to \infty} (P_n^{-4} - 1).$$

Consequently,

$$C(T) = \begin{cases} 0, & P_\infty = 0, \\ (P_\infty^{-4} - 1)^{-1/2}, & 0 < P_\infty < 1, \\ \infty, & P_\infty = 1. \end{cases} \tag{5.2}$$

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The depth-n experiment is exactly the channel P_{P_n}, so

$$m_n = 3P_n^2, \qquad a_n = \frac{3}{4}P_n.$$

This verifies the criterion on a ray directly, without Assumption 2.1.

For example, let $\lambda_j = 1 - (j+1)^{-2}$ for $j \geq 1$. Then

$$P_n = \frac{n+2}{2(n+1)} \longrightarrow \frac{1}{2}, \qquad C(T) = \frac{1}{\sqrt{15}}, \qquad a_\infty = \frac{3}{8}.$$

In contrast, simply replacing $\lambda^{2|v|}$ by R_v in the unweighted fixed-channel potential gives the divergent series $\sum_n P_n^{-4}$, and hence zero capacity on this ray. The factor $1-\lambda_e^4$ is therefore needed when parameters approach one. A perfectly transmitting ray has zero potential for every finite flow, so $C(T) = \infty$ and its spin observations reveal the root exactly.

5.2. **Zero channels.**

Corollary 5.1. *Assume Assumption 2.1, and allow $0 \leq \lambda_e \leq 1$. Let T_+ be the connected component of the root after deleting all zero-channel edges. Define capacity using (1.3)–(1.4) on T_+. Then Theorem 1.1, including its quantitative bounds, remains valid with $C(T_+)$ in place of $C(T)$.*

Proof. Realize each channel by copying the parent with probability λ_e and otherwise drawing a fresh uniform spin, using independent edge randomness. Across a zero edge the child is a fresh uniform spin independent of its parent. Group removed vertices by the first zero edge on their path from the root. Each resulting subtree depends only on its fresh initial spin and subsequent independent randomness. Jointly, all their observations are independent of both the root and all observations in T_+. Deleting them leaves the root posterior at every depth unchanged. Apply the theorem to the positive-channel tree T_+. If it is finite, distant levels are empty and conservation forces every flow to vanish. □

5.3. **The role of local finiteness.**

Local finiteness cannot be omitted without changing the statement. Give the root, for each $k \geq 1$, a disjoint child chain of length k, and take every channel to be perfect. There is no infinite ray. Conservation forces the flow on each finite chain to vanish, working backward from its leaf. Even with countably additive conservation at the root, every flow has mass zero. Yet each level n contains a spin equal to the root, from any chain of length at least n. Thus $m_n = 3$ and $a_n = 3/4$ for every n, although the capacity is zero.

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