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FACTOR-COUNT RESIDUES AND POISSON–DIRICHLET LIMITS FOR PRIME PREDECESSORS

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Abstract. Assuming specified analytic estimates for shifted primes and Dirichlet L-functions, we prove that $\Omega(p-1)$ modulo any fixed positive integer is asymptotically uniform and independent of the ranked logarithmic prime factor sizes, for primes in each fixed reduced arithmetic progression. The factor-size limit is the Poisson–Dirichlet distribution with parameter one. For parity, we also obtain joint independence from the squarefree indicator, with its progression-dependent density. This gives the Moree–Hommersom parity conjecture under the same assumptions. The proof combines marked correlations with a principal-term model that is compatible with small-prime multiplication and the extraction of prime arguments.

1. Introduction

For $n \geq 1$, let $\Omega(n)$ count prime factors with multiplicity. For each prime $p \geq 3$, list the prime factors of $p-1$ in decreasing order, with multiplicity, as $q_1(p), q_2(p), \ldots$, and put

$$V_i(p) = \frac{\log q_i(p)}{\log(p-1)}.$$

After the last prime factor, set $V_i(p) = 0$. The sequence $V(p)$ is a decreasing partition of one. Let $U_1, U_2, \ldots$ be independent uniform variables on $(0,1)$, and define

$$T_i = U_i \prod_{j<i}(1-U_j).$$

The sequence (T_i) has the GEM(1) distribution. Its decreasing rearrangement $L = (L_1, L_2, \ldots)$ has the Poisson–Dirichlet distribution PD(1). We write $\pi(X; M, b)$ for the number of primes at most X congruent to b modulo M.

The distribution of $V(p)$ alone does not determine the parity of $\Omega(p-1)$, or its residue modulo a larger integer. Small prime factors can change that residue without producing a detectable change in finitely many normalized factor sizes. We therefore prove cancellation of each nontrivial root-of-unity weight against marked factor statistics before passing to a probability limit.

Assumption 1.1 (Analytic inputs)**.** Assume the graph-transference, comparison, long-prime, and positive extraction estimates of [1, 2], the scalar marked-coefficient estimate of [3], and the zero-free half-plane $\operatorname{Re} s > 7/8$ for Dirichlet L-functions asserted in [4], with the principal pole at $s = 1$ retained. The precise estimates and their uniformity conditions are stated in Section 2; their source locations and fixed versions are specified in Appendix A.

The fixed-progression Poisson–Dirichlet law is proved below as Proposition 6.1; it is not inferred by conditioning the unrestricted source law. Its progression argument is an explicit dependency of both the residue theorem and the squarefree refinement.

Theorem 1.2 (Factor-count residues)**.** *Assume Assumption 1.1. Fix integers $M, m, k \geq 1$, a reduced class $b \pmod M$, a residue $a \pmod m$, and a continuous function $H : [0,1]^k \to \mathbb{C}$. Then*

$$\frac{1}{\pi(X; M, b)} \sum_{\substack{3 \leq p \leq X \\ p \equiv b \ (M)}} \mathbf{1}_{\Omega(p-1) \equiv a \ (m)} H(V_1(p), \ldots, V_k(p)) \longrightarrow \frac{1}{m} \mathbb{E} H(L_1, \ldots, L_k). \tag{1.1}$$

The same limit holds for primes in $(x, 2x]$, with denominator $\pi(2x; M, b) - \pi(x; M, b)$.

Every displayed parameter is fixed before the real endpoint tends to infinity. The theorem gives no rate and no uniformity in growing moduli. Taking $H = 1$ gives uniform residues; dividing by their limiting probability $1/m$ gives the conditional factor-size law.

Write $\lambda(n) = (-1)^{\Omega(n)}$ and let μ be the Möbius function. For a prime q, set $e_q = v_q(M)$ and define

$$\rho_q(M,b) = \begin{cases} 1/[q(q-1)], & e_q = 0, \\ 1/q, & e_q = 1,\ b \equiv 1 \pmod q, \\ 0, & e_q = 1,\ b \not\equiv 1 \pmod q, \\ 1, & e_q \geq 2,\ b \equiv 1 \pmod{q^2}, \\ 0, & e_q \geq 2,\ b \not\equiv 1 \pmod{q^2}. \end{cases} \qquad C_{M,b} = \prod_q (1 - \rho_q(M,b)). \tag{1.2}$$

The product includes $q = 2$.

Theorem 1.3 (Parity and squarefreeness)**.** *Under Assumption 1.1, if $p \geq 3$ is uniform among the class primes in either sampling convention of Theorem 1.2, then*

$$(V_1(p), \dots, V_k(p), \lambda(p-1), \mu^2(p-1)) \Longrightarrow (L_1, \dots, L_k, S, B),$$

where S is a fair sign, B is Bernoulli with parameter $C_{M,b}$, and L, S, B are mutually independent. In particular, for $s \in \{-1, 1\}$,

$$\mathbb{E}[\mu(p-1) H(V_1(p), \dots, V_k(p))] \longrightarrow 0, \tag{1.3}$$
$$\mathbb{E}[\mathbf{1}_{\mu(p-1)=s} H(V_1(p), \dots, V_k(p))] \longrightarrow \frac{C_{M,b}}{2} \mathbb{E} H(L_1, \dots, L_k).$$

The notation $\mathbb{E}$ on the left denotes the finite prime average. If a square is forced by the class, $C_{M,b} = 0$ and the Bernoulli variable is zero. Conditioning on squarefreeness or on a Möbius sign requires $C_{M,b} > 0$. For $M = 1$, each Möbius sign has density $\frac{1}{2} \prod_q (1 - 1/[q(q-1)])$ relative to the primes.

For a finite set $\mathcal{S}$ of primes, define

$$\mu_{\mathcal{S}}(n) = \mu\left(n \Big/ \prod_{q \in \mathcal{S}} q^{v_q(n)} \right).$$

Thus square factors in $\mathcal{S}$ do not force this function to vanish.

Corollary 1.4 (Prescribed valuations)**.** *Under Assumption 1.1, for every finite $\mathcal{S}$ and fixed nonnegative integers e_q, $q \in \mathcal{S}$,*

$$\sum_{\substack{p \leq X \\ v_q(p-1) = e_q \ (q \in \mathcal{S})}} \mu_{\mathcal{S}}(p-1) = o(\pi(X)).$$

This is the parity assertion of Moree and Hommersom [10, Conjecture 1]*, under the stated source assumptions.*

1.1. **Related work and proof outline.** The ordinary-integer distribution results of Pintz [11] and Granville and Shao [8] supply progression estimates used below. They do not restrict the averaging variable to prime predecessors. Lichtman [9] proves cancellation averaged over a growing range of shifts. Carella [6, Theorems 1.1–1.2] states shifted-prime cancellation subject to a separate distribution hypothesis. The joint law in (1.1) concerns the fixed shift -1.

The factor-count residue is recovered by finite Fourier inversion. The analytic problem is to prove cancellation for $f_z(n) = z^{\Omega(n)}$, for each nontrivial root of unity z. For $z = -1$, the ordinary mean has arbitrary logarithmic cancellation. For nonreal z, the Selberg–Delange expansion has a

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nonzero principal term. Sections 4 and 5 construct and propagate that term. Section 6 derives the joint laws from the resulting interior factor estimates.

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2. Marked sums and analytic inputs

Put $L = \log x$ and $\mathrm{e}(t) = \exp(2\pi i t)$. A constant denoted by C may change between estimates. Every exponent of L is fixed before x tends to infinity. The notation $O_A(L^{-A})$ means that each fixed accuracy $A > 0$ is available, after the stated choices of parameters.

2.1. **Long slots, inert bands, and active groups.** Fix $d \ge 1$ and $\varepsilon > 0$. Let $I_1, \dots, I_d$ be intervals of primes whose lower endpoints are at least x^ε and whose upper endpoints have product at most $x^{1-\varepsilon}$. Tuples are ordered and their numerical prime entries are distinct. Write

$$T_x(n) = \sum_{\boldsymbol{\ell}} \mathbf{1}_{\ell|n}, \qquad \ell = \ell_1 \cdots \ell_d, \qquad C_x = \sum_{\boldsymbol{\ell}} \frac{1}{\ell}. \tag{2.1}$$

On each fixed range $n \le x^C$, $T_x(n) = O_{C,d,\varepsilon}(1)$. Also $C_x = O_{d,\varepsilon}(1)$ by the harmonic prime estimate.

An inert array consists of K disjoint full prime intervals $[\exp(L^{a_i}), \exp(2L^{a_i}))$, with fixed $.1 < a_i < .2$. For damping $0 < q < 1$, its weight is

$$\mathcal{W}(n) = \prod_{i=1}^{K} q^{\omega_i(n)-1} \frac{\omega_i(n)}{V_i}, \quad \omega_i(n) = \sum_{p \in \mathcal{P}_i} \mathbf{1}_{p|n}, \quad V_i = \sum_{p \in \mathcal{P}_i} \frac{1}{p}. \tag{2.2}$$

A factor is zero when $\omega_i(n) = 0$. For fixed K, q, the weight is bounded. Its independent divisor-model mean $m_{\mathcal{W}}$ is obtained by replacing the divisibility indicators by independent Bernoulli variables with means $1/p$. The source gives $0 < c_K \le m_{\mathcal{W}} \le C_K$ and $\mathbb{E}_{\mathrm{div}} \mathcal{W}^2 \le C_K$, uniformly over the fixed exponents.

Independently, choose disjoint active groups $\mathcal{P}_g$, in two families $\mathcal{G}_s, \mathcal{G}_b$. Each family has between fixed positive multiples of $\log L$ groups, and

$$\mathcal{P}_g \subset [\exp(L^{.27}), \exp(L^{.36})], \qquad g \in \mathcal{G}_s,$$
$$\mathcal{P}_g \subset [\exp(L^{.39}), \exp(L^{.46})], \qquad g \in \mathcal{G}_b.$$

Their harmonic masses lie in a fixed compact subinterval of $(0, \infty)$. Every big group is a full prime interval. At least one small group is also a full interval, as needed for fixed-modulus mixing. The extraction uses full intervals for all candidate groups.

Let n_* denote n after removal of all powers of active primes. For $\mathbf{t} = (t_g)$ and a bounded function b of ordered distinct represented divisor lists, set

$$W_{\mathbf{t}}^{b}(n) = q_0^{\omega_P(n) - \sum_g t_g} \prod_g V_g^{-t_g} \sum_{\text{represented lists at } n} b(\mathbf{p}), \qquad 0 < q_0 < 1. \tag{2.3}$$

Here $\omega_P(n) = \sum_g \omega_g(n)$ counts distinct active prime divisors, and $V_g = \sum_{p \in \mathcal{P}_g} 1/p$. Write W_1 when $t_g = 1$ for all g and $b = 1$. For fixed bounded list lengths and $|b| \le 1$, these weights are at most a fixed power of L. The second endpoint will have the form $G(n) = G(n_*)$ with $|G(n)| \le L^C \tau(n_*)^C$.

2.2. **The source estimates used at the two endpoints.** We use the general physical pairing in [1, Section 6], rather than extending its final centered correlation theorem by substitution. The following description fixes its measures and the parts of its conclusion that enter the proof.

For a sufficiently large fixed integer J, a physical state at n has $J+1$ ordered distinct prime divisors from each active group. Its list mass is $\prod_g V_g^{-(J+1)}$, with counting measure in n. In a pattern ν, the source and target share $J - |C_g|$ labels in group g. Small groups have $C_g = \varnothing$; big

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groups have $|C_g| \le m_0$ for a fixed m_0. Let D_R be the shared product and D_C the product of the auxiliary free labels. Free labels have independent laws $1/(V_g p)$. The physical shift is $D_R D_C$. The pattern kernels depend only on free big labels and unshared big endpoint lists. The family has total kernel mass $L^{O(1)}$, with exponent independent of J.

The residual ideal-family theorem supplies a raw pattern $D_C = 1$ and signed comparison patterns. Its ideal operator has arbitrarily small logarithmic norm. The general transfer theorem then bounds the whole physical pairing for a first vector independent of its lists at fixed n, with the source supremum bound, and endpoint norms $\le X^{1/2} L^{C_3}$ on a position dyad $[X, 2X)$. The exponent C_3 must be independent of J. Both endpoints retain the extra half damping $q_0^{(\omega_P(n)-(J+1)|P|)/2}$, which is at most one on represented states. If the first core is bounded and the second has the divisor bound above, the required norm estimate follows from

$$\prod_g V_g^{-(J+1)} \sum_{\text{lists}} \frac{1}{\prod p} \le 1 \tag{2.4}$$

and a fixed divisor moment. Conjugate the first core in the physical vector to cancel the conjugation in the Hilbert-space pairing.

After inserting $\Psi(n/(xD_RD_C))/n$, restoring its full Mellin integral, and summing all position dyads, put

$$n = D_R w, \quad n' = D_R(w + D_C), \qquad \frac{\Psi(n/(xD_RD_C))}{n} = \frac{1}{D_R w} \Psi\left(\frac{w}{xD_C}\right). \tag{2.5}$$

The remaining cutoff has no shared-label dependence. The factor D_R^{-1} makes the shared-label measures harmonic probabilities. The source's collision and Mellin-tail errors save every fixed power of L, also after bounded multipliers at the endpoints.

Every comparison has two nonempty free big products, at scales U_1, U_2. Write

$$H' = U_1U_2, \quad Y = xH', \quad H = H'/L^{C_6}, \quad T_0 = LY/H = xL^{C_6+1}, \qquad U_i \ge \exp(L^{.39} - O(1)). \tag{2.6}$$

Expanding the source kernels reduces a comparison to bounded separate tests on the two free tuples, with a total fixed logarithmic cost. Its additive multiplier is small off rational arcs of denominator $\le L^{C_6}$ and width H^{-1}. The source's local Mellin estimate reduces the first-endpoint energy on these arcs to

$$\int_{|t|\le T_0} \left| Y^{-1} \sum_n \mathcal{U}(n) \mathrm{e}(hn/k) n^{it} \right|^2 \mathrm{d}t \ll_A L^{-A}, \qquad k \le L^{C_6}. \tag{2.7}$$

Here $\mathcal{U}$ has a smooth cutoff at Y, a logarithmic-sized twist, the proposed first core, and a factor $\mathcal{W}W_{\mathbf{t}}^b$ whose test b uses only big marks. The second endpoint needs only $\sum |\mathcal{V}(n)|^2 \ll YL^C$. The estimates and kernel expansions in this paragraph are the input package from [1, Section 6]; we prove (2.7) for the new cores below.

The truncated divisor model for $\mathcal{W}W_{\mathbf{t}}^b$ has the form

$$\sum_{a|n} A_a + E(n), \quad a \le \exp(O(L^{.47})), \quad \sum_a \frac{|A_a|}{a} \le L^C, \quad \sum_{n \asymp R} |E(n)|^2 \ll_N RL^{-N}. \tag{2.8}$$

It permits a zero list length when a small mark has been removed. The constants are uniform in bounded big-mark tests.

We also use two explicit inputs from [1, Appendix A]. For fixed $0 < \tau < \eta < 1$ and $A, C > 0$, there is B_0 such that

$$\left| \sum_{p \in I} \chi(p) p^{-1+it} \right| \ll L^{-A} \tag{2.9}$$

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for $I \subset [N, 2N]$, $x^{\tau}/2 \le N \le x^{\eta}$, $\mathrm{mod}(\chi) \le L^{C}$, and $L^{B_0} \le |t| \le x^2$. For some absolute C_*, the second estimate is

$$\left| \sum_{\substack{n \in I \\ n \equiv a \ (k)}} n^{it} \right| \ll \frac{N'}{k} \exp\left(-\frac{L}{(\log L)^{C_*}} \right), \tag{2.10}$$

where $I \subset [N', 2N']$, $k \le L^{C}$, $\exp(L/(\log L)^2) \le N' \le 2x^5$, and $\exp(L/(2(\log L)^2)) \le |t| \le 4x^3$. The residue a need not be a unit. The subinterval uniformity in both estimates allows partial summation against the smooth amplitudes below.

2.3. **The scalar determinant and positive extraction inputs.** The coefficient criterion in [3, Section 8] applies to rough dyadic coefficients α_m, β_n, bounded by fixed powers of L, on positive-power scales $H_m H_n \asymp x$. It requires

$$H_m^{-1} \left| \sum_{m} \alpha_m \chi(m) m^{it} \right| \ll_A L^{-A} \tag{2.11}$$

for every fixed logarithmic modulus range, every fixed accuracy, and $|t| \le 2H_m H_n L^{B}$. The scalar estimate in its proof bounds the expanded Cauchy square after $mn - 1 = ah$, with a a product of K selected inert primes. An invariant bounded factor of h lies outside that square. The source prime-minus-rough coefficient meets (2.11). Multiplication by a character of a fixed modulus M replaces the test modulus by its least common multiple with M, which is still logarithmic.

The extraction in [1, Section 7] uses a nonnegative block sieve, prime-to-rough telescoping, and a partition into independent conditional coin outcomes. The block sieve has relative error $O(e^{-h})$ at even depth h and uses summed progression remainders only up to z^{4h+2} when the sieve limit is z. For the independent divisor model, its local densities are $1/r$ at allowed primes. The rough proxy is

$$a(m) = \frac{\mathbf{1}_{m > x^{\gamma}} \mathbf{1}_{P^-(m) > W}}{V(W) \log m}, \qquad W = \exp(L^{.24}), \quad V(W) = \prod_{p \le W} (1 - 1/p).$$

The source permits removal of all candidate prime parts from its thresholds and logarithms at absolute cost $o(x/L)$. It also gives a total positive proxy mass $O_{j,\gamma}(x/L)$ for each fixed number j of factors. Section 5 states the allocation identity and carries out the needed new recombination. These imported estimates do not assert cancellation of $f_z(p-1)$.

3. Fixed progressions and the Liouville weight

3.1. **Two residue conditions before division by shared labels.** Fix M and a residue $r \pmod M$, which may be a nonunit. Multiply the first physical vector by $\mathbf{1}_{n \equiv r \ (M)}$ and the second by $\mathbf{1}_{n' \equiv r+1 \ (M)}$. Their norms decrease. The first vector remains independent of its lists at a fixed position. Thus the general pairing still bounds the entire residual family.

For a full small prime interval $\mathcal{P}_g$, the harmonic law $1/(V_g p)$ is uniform on $(\mathbb{Z}/M\mathbb{Z})^{\times}$ to any fixed negative power of L. Partial summation of the fixed-modulus prime estimate proves this, since $\log p \ge L^{.27}$ and V_g is bounded below. One shared label from this group suffices. Conditional on every other label in the independent-label model, multiplication by their product permutes the units. The collision errors described in Section 2 remain negligible.

Perform this averaging only after the dyads and Mellin integral have been restored in (2.5). For a unit $d \pmod M$, the exact identity is

$$\frac{1}{\varphi(M)} \sum_{t \in (\mathbb{Z}/M\mathbb{Z})^{\times}} \mathbf{1}_{tw \equiv r \ (M)} \mathbf{1}_{t(w+d) \equiv r+1 \ (M)} = \frac{1}{\varphi(M)} \mathbf{1}_{w \equiv rd \ (M)}. \tag{3.1}$$

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Subtracting the two congruences forces $td \equiv 1$. The unique possible $t = d^{-1}$ satisfies the first congruence exactly when $w \equiv rd$. There is no division by r or w. The raw pattern has $d = D_C = 1$. Each comparison acquires $w \equiv rD_C \pmod M$.

To bound those comparisons, first split each free product into its fixed unit residues modulo M. The remaining condition on w is a fixed residue. With the Fourier convention $\widehat{U}(\theta) = \sum_n U(n)\mathrm{e}(\theta n)$,

$$\widehat{\mathbf{1}_{n\equiv s\ (M)}U}(\theta) = \frac{1}{M}\sum_{j \bmod M} \mathrm{e}(-js/M)\widehat{U}(\theta + j/M).$$

Its squared energy is bounded by the corresponding average of the squared energies. A rational center with denominator k moves to one with denominator at most Mk. Increasing the fixed logarithmic denominator and width exponents covers all these arcs. In (2.6), H' grows faster than every fixed power of L times a small-prime scale, so the new time cutoff is still admissible. The bounded tests on the free products preserve the bilinear multiplier estimate.

Apply this argument to the source centered endpoint $\mathcal{W}(T_x - C_x)$. The source's local energy estimate already holds at every fixed logarithmic denominator and time exponent. It yields

$$\sum_{w\equiv r\ (M)} \frac{\Psi(w/x)}{w}\mathcal{W}(w)(T_x(w) - C_x)G(w+1)W_1(w)W_1(w+1) \ll_A L^{-A}. \tag{3.2}$$

We will use this to prove the unsigned progression law. Placing the residue condition inside G would violate $G(n) = G(n_*)$; the two physical residue conditions avoid that problem.

3.2. **Ordinary-integer cancellation.** The classical Liouville estimate, in the reduced-residue form recorded by Pintz [11, equation (1.5)], is

$$\sum_{k\le R^\theta} \max_{(a,k)=1} \left| \sum_{\substack{n\le R\\ n\equiv a\ (k)}} \lambda(n) \right| \ll_{A,\theta} R(\log R)^{-A}, \qquad \theta < \tfrac{1}{2}. \tag{3.3}$$

All parameters are fixed. Smooth compactly supported dyadic weights follow by partial summation, with the sum over moduli inside the integral of absolute cumulative sums. No supremum over endpoints is inserted into (3.3).

If a residue $a \pmod k$ is a nonunit, set $g = (a,k)$ and write $n = gv$. Then $v \equiv a/g \pmod{k/g}$ is a reduced class and $\lambda(gv) = \lambda(g)\lambda(v)$. For $R \ge x^\delta$, $k \le L^C$, and $|t| \le L^B$, this gives

$$\frac{1}{R}\sum_{n\equiv a\ (k)} \lambda(n)\psi(n/R)n^{it} \ll_A L^{-A}. \tag{3.4}$$

The test ψ has fixed compact support in $(0,\infty)$, with fixed logarithmic derivative costs. The reduced scale R/g is still a positive power of x; increasing the accuracy in (3.3) pays every displayed logarithmic cost.

3.3. **A signed correlation with a nonempty tuple.**

Lemma 3.1. *For the marked data of Section 2, fixed M, r, and $\Psi \in C_c^\infty((0,\infty))$,*

$$\sum_{w\equiv r\ (M)} \frac{\Psi(w/x)}{w}\lambda(w)\mathcal{W}(w)T_x(w)G(w+1)W_1(w)W_1(w+1) \ll_A L^{-A}. \tag{3.5}$$

The tuple dimension satisfies $d \ge 1$.

Proof. Insert $\lambda(n)$ in the first physical vector. Its norm and list independence are unchanged. If s_P is the number of active groups and $c_\nu = \sum_g |C_g|$, then

$$\Omega(D_R) = Js_P - c_\nu, \qquad \lambda(D_R w) = (-1)^{Js_P}(-1)^{c_\nu}\lambda(w).$$

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These identities count repeated auxiliary labels with multiplicity. Divide out the common sign $(-1)^{Js_P}$. The transference result and (3.1) give the raw correlation plus comparisons multiplied by $(-1)^{c_\nu}$, up to $O_A(L^{-A})$. Each comparison still needs a separate bound.

At low Mellin times $|t| \le L^B$, expand the tuple and the divisor model (2.8), writing $w = \ell a v$. The scale of v is $R = Y/(\ell a) \ge x^{\varepsilon - o(1)}$. Complete multiplicativity separates $\lambda(\ell a)$, and splitting the additive phase into residue classes permits (3.4). The total normalized error is bounded using $\sum_\ell 1/\ell = O(1)$ and $\sum |A_a|/a \le L^C$. Cauchy's inequality controls the divisor-model error without changing its square norm. Choose the accuracy after B so that the integral over these low times also saves every required power of L.

At high times remove one small mark and one long prime slot, giving $w = p_s p_\ell v$. Then $\lambda(w) = \lambda(v)$, so the small- and long-prime polynomials are unchanged up to constant signs. Their residual coefficient has the same square norm as the original coefficient. Small-mark squares and repeated numerical long-slot primes have square errors $O(YL^C \exp(-L^{.27}))$ and $O(YL^C x^{-\varepsilon})$. Ordinary Dirichlet mean square bounds dispose of these errors on the whole time interval.

The factorial estimate in [1, Section 6] places the times when the small-prime polynomial exceeds a prescribed negative power of L inside at most $\exp(O(L^{.9}))$ unit intervals. Outside these intervals, ordinary mean square on the collapsed scale Y/P_s applies because $T_0/(Y/P_s) = LP_s/H \to 0$. Here P_s denotes the small-prime scale, so $P_s \le \exp(L^{.36})$. On the exceptional intervals, use (2.9) and the source sparse Gram estimate for the remaining coefficients. That estimate is coefficient-independent: its matrix entries are the unweighted sums $\sum_{v \asymp R} v^{i(t_i - t_j)}$. Multiplying the tested coefficient by $\lambda(v)$ preserves its square norm. The long-prime scale lies between $x^\varepsilon/2$ and $x^{1-\varepsilon}$; the residual scale is at least $x^{\varepsilon - o(1)}$. All retained times are below x^2 after fixed logarithmic twists. These are the source estimate's ranges. Choose B and the long-prime accuracy after the fixed comparison costs.

This proves (2.7). The Fourier shifts for the fixed residue condition use the same bounds with enlarged fixed exponents. The local Mellin and multiplier inputs now bound every comparison. Subtract them from the transferred residual to obtain (3.5). □

The nonempty tuple supplies the long prime in this proof. We have not proved a signed constant-endpoint estimate by this method. The total Liouville mean will follow instead from size-biased sampling.

3.4. **Progression remainders and scalar replacement.** Fix $0 < c_0 < \varepsilon/8$ and a smooth nonnegative compact cutoff Φ. For H equal to a fixed inert weight, a product of two such weights, a constant, or a bounded linear combination of these, the source represented-divisor expansion and CRT give

$$\sum_{\substack{r \le x^{c_0} \\ (r,M)=1}} \left| \sum_{\substack{r \mid u+1 \\ u+1 \equiv b \ (M)}} \Phi(u/x) H(u) T_x(u) - \frac{x}{rM} \left(\int \Phi \right) C_x \mathbb{E}_{\mathrm{div}} H \right| \ll_A x L^{-A}. \tag{3.6}$$

The statement also holds with T_x, C_x replaced by $1, 1$. Indeed every retained tuple/divisor product is eventually coprime to fixed M. A divisor sharing a prime with r gives no solutions; otherwise CRT gives one class modulo rM. The source comparison of the forbidden inert primes with its independent model has exponentially small error. The total rounding error is $x^{1-\varepsilon+c_0+o(1)}$. Divisors with $(r, M) > 1$ contribute zero.

The corresponding signed remainder is

$$\sum_{r \le x^{c_0}} \left| \sum_{\substack{r \mid u+1 \\ u+1 \equiv b \ (M)}} \Phi(u/x) H(u) \lambda(u) T_x(u) \right| \ll_A x L^{-A}. \tag{3.7}$$

To see this, write $u = \ell av$ in each retained divisor term. When compatible, the congruences specify a class of v modulo rM. Its gcd with rM is the fixed $g = (b-1, M)$. After $v = gt$, the class is reduced modulo rM/g. The moduli are distinct as r varies and are less than $(x/(\ell ag))^{1/3}$ for large x. Apply (3.3) at that residual scale and sum with the reciprocal tuple and divisor coefficients. For the omitted terms use the positive first-omitted Bonferroni divisor envelope and CRT. Its factorial reciprocal tail saves every logarithmic power, and its rounding error has the bound above. This direct progression argument avoids multiplying an L^2 error by x^{c_0}.

Finally, the scalar replacement extends to all unit-modulus weights $f_z(n) = z^{\Omega(n)}$ used here. Selecting one prime from each of K inert bands and writing $mn - 1 = ah$ gives

$$f_z(mn-1)F_0(mn-1) = z^K f_z(h)F_0(h) \tag{3.8}$$

for an invariant bounded F_0. The factor on the right lies outside the original Cauchy square and has modulus at most one. The scalar coefficients α, β are unchanged, so the assumed coefficient criterion applies. The selected-prime-square error is absolute. The product class is imposed by

$$\mathbf{1}_{mn\equiv b\ (M)} = \frac{1}{\varphi(M)} \sum_{\chi \bmod M} \overline{\chi(b)}\chi(m)\chi(n).$$

The character factors preserve (2.11), with logarithmic test moduli enlarged by a fixed factor. No Liouville or complex weight is inserted into the prime-minus-rough discrepancy coefficient.

4. The principal term for a nonreal mode

Fix a nonreal root of unity z, and put $f_z(n) = z^{\Omega(n)}$. The proof below treats each such z separately. All constants may depend on z. In particular, no estimate uniform as $z \to 1$ is claimed.

4.1. **Classical distribution inputs.** For $\operatorname{Re} s > 1$,

$$\sum_{n\ge 1} \frac{f_z(n)}{n^s} = \zeta(s)^z G_z(s), \qquad G_z(s) = \prod_p \frac{(1-p^{-s})^z}{1 - zp^{-s}}. \tag{4.1}$$

Branches are continued from this half-plane. The product G_z is analytic near 1. The Selberg–Delange theorem of de la Bretèche and Tenenbaum [5, Theorem 1.2 and equation (1.10)] gives a fixed-function expansion to arbitrary fixed logarithmic precision. Its hypotheses hold since $f_z(p) = z$, $|f_z| = 1$, and the higher prime-power sums converge with any fixed exponent $\sigma > 1/2$.

Set $A_z(s) = ((s-1)\zeta(s))^z G_z(s)$. The cumulative coefficients are those of $A_z(s)/s$, divided by $\Gamma(z-j)$. The smooth density coefficients are those of $A_z(s)$. This follows by Stieltjes partial summation of a finite cumulative expansion, with its remainder bounded in absolute value. In detail, if $A_z(1+t) = \sum a_j t^j$ and $A_z(1+t)/(1+t) = \sum c_j t^j$, then $a_j = c_j + c_{j-1}$. Differentiating the finite terms gives

$$\frac{c_j}{\Gamma(z-j)} + \frac{(z-j)c_{j-1}}{\Gamma(z-j+1)} = \frac{a_j}{\Gamma(z-j)}.$$

There is no differentiation of an unspecified error.

We also require the centered progression estimate

$$\sum_{q\le R^\theta} \max_{(a,q)=1} \left| \sum_{\substack{n\le R\\ n\equiv a\ (q)}} f_z(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\le R\\ (n,q)=1}} f_z(n) \right| \ll_{z,A,\theta} R(\log R)^{-A}, \qquad \theta < \tfrac{1}{2}. \tag{4.2}$$

Here is the verification of the hypotheses needed to apply Granville–Shao [8, Theorem 1.1(b)]. For a nonprincipal character χ, the twisted Dirichlet series is $L(s,\chi)^z$ times a product absolutely convergent in $\operatorname{Re} s > 1/2$. Assumption 1.1 gives a zero-free half-plane $\operatorname{Re} s > 7/8$. On each closed smaller half-plane, Borel–Carathéodory applied to discs centered at $2 + it$ bounds the analytic logarithm of L by $O(\log(q(2+|t|)))$. Polynomial bounds for L follow from character periodicity

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and partial summation; the Euler series fixes the logarithm at the centers. The discs avoid zeros and poles. Extra Euler factors of imprimitive characters are nonzero there. Thus the twisted series has polynomial growth.

Smooth Mellin inversion can be moved to $\operatorname{Re} s = 15/16$. For $q \le (\log R)^D$, it gives arbitrary logarithmic saving after choosing a sufficient transform decay order. Smooth majorants with transition width $R(\log R)^{-K}$ remove the cutoff with an error smaller than any prescribed negative logarithmic power. Character orthogonality gives the centered Siegel–Walfisz criterion. For larger q, choose D large and use the elementary bounds $R/q + 1$ and $R/\varphi(q)$, treating $q > R$ separately.

The logarithmic derivative coefficients satisfy $\Lambda_{f_z}(p^j) = z^j \log p$. Its convolution inverse is μf_z, whose logarithmic derivative coefficients are the negatives of these. Both functions belong to the class required by [8]. On primes, $f_z = z\mathbf{1}_{\mathbb{P}}$, so classical prime Bombieri–Vinogradov gives the other hypothesis of Theorem 1.1(b). This proves (4.2). Smooth versions follow with the sum over moduli inside partial summation and a strict exponent margin at every comparable scale. The principal branch at $s = 1$ has not been removed by this argument.

4.2. **Definition and exact covariance.** Put $\rho = 23/50$, $y = \exp(L^\rho)$, and

$$n_y = \prod_{p \le y} p^{v_p(n)}, \qquad V_y = \prod_{p \le y} (1 - 1/p), \qquad P_{z,y}(s) = \prod_{p \le y} (1 - zp^{-s}).$$

For a sufficiently large fixed integer N, define

$$b_{z,y,N}(v) = \sum_{j=0}^{N-1} \frac{(A_z P_{z,y})^{(j)}(1)}{j!\Gamma(z-j)} (\log v)^{z-1-j}, \qquad v > 1. \tag{4.3}$$

Choose a fixed smooth real χ, between zero and one, equal to zero on $(-\infty, \varepsilon/4]$ and one on $[\varepsilon/2, \infty)$. Set $\beta(v) = \chi(\log v/L) b_{z,y,N}(v)$ for $v > 1$ and zero otherwise. Define the principal term on every positive integer by

$$\mathcal{B}_x(n) = \frac{f_z(n_y)}{V_y}\, z^d \sum_{\boldsymbol{\ell}} \frac{1}{\ell} \beta\left(\frac{n}{n_y \ell}\right). \tag{4.4}$$

This definition uses all primes at most y, not an active subset. The same function therefore occurs for every active set used later.

Lemma 4.1. *Let $\delta_z = 1 - \operatorname{Re} z > 0$. On each fixed range $n \le x^C$,*

$$|\mathcal{B}_x(n)| \ll_{N,z,d,\varepsilon,C} L^{-(1-\rho)\delta_z}. \tag{4.5}$$

For every y-smooth integer a, $\mathcal{B}_x(an) = f_z(a)\mathcal{B}_x(n)$. The logarithmic exponents in bounds for any fixed number of smooth derivatives can be chosen independently of fixed N; their constants and thresholds may depend on N.

Proof. The identity $(an)_y = an_y$ gives exact covariance, also when a and n share prime factors. For each fixed $j \ge 1$,

$$|(\log P_{z,y})^{(j)}(1)| \ll_{j,z} (\log y)^j, \qquad |(A_z P_{z,y})^{(j)}(1)| \ll_{j,z} |P_{z,y}(1)| (\log y)^j.$$

The second bound also holds for $j = 0$. These estimates follow by differentiation of the finite Euler product and $\sum_{p \le y} (\log p)^j/p \ll_j (\log y)^j$. On the cutoff support, successive density terms have a factor $O_{j,z,\varepsilon}(L^{\rho-1})$. Also

$$\frac{|P_{z,y}(1)|}{V_y} = \exp\left(\delta_z \sum_{p \le y} \frac{1}{p} + O_z(1)\right) \asymp_z L^{\rho\delta_z}.$$

Multiply this by $L^{\operatorname{Re} z - 1}$ and the bounded harmonic tuple sum. This proves (4.5). The same calculation applies to a fixed number of logarithmic derivatives. The finite sequence of decreasing powers $L^{-(1-\rho)j}$ absorbs their constants for large x depending on N. □

The leading coefficient of (4.3) is $G_z(1)P_{z,y}(1)/\Gamma(z)$. Taking absolute values of all Euler coefficients before summing would lose the bound just proved.

4.3. **Truncations with square and progression bounds.** Let $D_0 = \exp(L^{1/2})$ and $h_0 = \lfloor L^{.01} \rfloor$. For $\eta = 1/\log y$, Rankin's inequality gives

$$\sum_{P^+(a)\le y} \frac{a^\eta}{a} \ll L^C, \qquad \sum_{\substack{P^+(a)\le y \\ a>D_0/y}} \frac{1}{a} \ll L^C \exp(-\tfrac{1}{2}L^{.04}). \tag{4.6}$$

If $n_y > D_0$, multiplying its prime factors one at a time finds a divisor in $(D_0/y, D_0]$. This gives a positive short-divisor majorant for the discarded integers. Fixed local divisor moments remain logarithmic, since their local Euler factors are $1+O(1/p)+O(1/p^2)$. Cauchy's inequality then gives arbitrary logarithmic square savings as well. On a represented physical list with product P, the Rankin weight adds at most $P^\eta \le L^{O_J(1)}$. The saving in (4.6) absorbs that cost after J is fixed.

For roughness use the finite Bonferroni expansion

$$\mathbf{1}_{P^-(v)>y} = \sum_{\substack{e\mid v, P^+(e)\le y \\ \mu^2(e)=1, \omega(e)\le h_0}} \mu(e) + E(v), \qquad |E(v)| \le \binom{\omega_{\le y}(v)}{h_0+1}. \tag{4.7}$$

The last expression is a positive divisor sum. Its divisors are $\le \exp(O(L^{.47}))$, and their reciprocal mass is at most $(C\log L)^{h_0+1}/(h_0+1)!$. The retained absolute reciprocal mass is $L^{O(1)}$.

For a tuple term $u = \ell v$, a divisor condition $a \mid v$ and compatible congruences $r \mid u+1$, $u+1 \equiv b \pmod M$ give $O_M(x/(\ell r a)+1)$ integers on the support. There are $O_d(x^{1-\varepsilon})$ tuples. Every fixed product of the short-divisor supports above has ordinary coefficient mass $x^{o(1)}$. Summing the rounding errors over $r \le x^{c_0}$ therefore costs

$$x^{1-\varepsilon+c_0+o(1)} = O_A(xL^{-A}). \tag{4.8}$$

For model terms the tuple is harmonic rather than a divisibility condition, and the rounding cost is only $x^{c_0+o(1)}$.

For square bounds, sum the omitted roughness terms over the possible smooth parts before applying a norm inequality. Apart from a fixed logarithmic factor and the bounded harmonic tuple sum, the pointwise error is at most

$$\tau(n_y)2^{\omega_{\le y}(n)}\mathbf{1}_{\omega_{\le y}(n)\ge h_0}.$$

Squaring, using $4^w\mathbf{1}_{w\ge h_0} \le 2^{-h_0}8^w$, and applying a fixed divisor moment gives $O(RL^C2^{-h_0})$ on a dyad of scale R. This is the aggregate square bound needed below. Summing separate norms with a factor $\sum_{a\le D_0} a^{-1/2}$ would not suffice.

4.4. **Matching the local principal terms.**

Lemma 4.2 (Rough density)**.** *Fix a positive residual exponent ε_1 and fixed logarithmic modulus, time, and derivative bounds. For $R \in [x^{\varepsilon_1}, x^C]$, $k \le L^C$, $(a,k)=1$, $|t| \le L^B$, and a smooth dyadic test ψ, every sufficiently large fixed N gives*

$$\frac{1}{R}\sum_{\substack{v\equiv a\ (k) \\ P^-(v)>y}} f_z(v)\psi(v/R)(v/R)^{it} = \frac{1}{\varphi(k)}\int \psi(u)u^{it}b_{z,y,N}(Ru)\,\mathrm{d}u + O_A(L^{-A}). \tag{4.9}$$

For unweighted rough integers the density is $V_y/\varphi(k)$, with the same accuracy and smoothness bounds.

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Proof. Every prime of k is at most y for large x. Use (4.7). The nonprincipal character sums save every power of L by the smooth Siegel–Walfisz estimate proved above, on scales $R/e \ge x^{\varepsilon_1-o(1)}$. The retained reciprocal mass and the error envelope pay the fixed modulus and time costs.

For the principal character, apply the fixed-function Selberg–Delange expansion to each residual scale R/e. Taylor-expand $\log(Ru/e)$ around $\log(Ru)$. Since $\log e = O(L^{.47})$, each further Taylor order gains a factor $L^{-.53}$ before fixed costs. The restored moments are

$$\sum_{P^+(e)\le y} \frac{\mu(e)f_z(e)}{e}(-\log e)^j = P_{z,y}^{(j)}(1).$$

Restoring the omitted degrees in each fixed moment costs at most

$$2^{-h_0} \sum_{e\mid \prod_{p\le y} p} \frac{2^{\omega(e)}(\log e)^j}{e} \ll_j 2^{-h_0} L^{C_j}.$$

The derivative product rule and

$$\frac{1}{\Gamma(z-j)}\binom{z-1-j}{r} = \frac{1}{r!\Gamma(z-j-r)}$$

give (4.3). Choosing sufficiently many fixed terms gives the desired error; the bounds in Lemma 4.1 permit any fixed N beyond a sufficient threshold. The principal character is one on every rough integer, so its contribution is divided by $\varphi(k)$. For the unweighted case, CRT in (4.7) gives density $k^{-1}\prod_{p\le y, p\nmid k}(1-1/p) = V_y/\varphi(k)$. □

Lemma 4.3 (Local matching)**.** *Let $H_x(n) = H_x(n_y)$, with $|H_x| \le L^C$. At $Y = x^{1+o(1)}$, uniformly for logarithmic k, t and the permitted smooth test,*

$$\frac{1}{Y}\sum_n \psi(n/Y)(n/Y)^{it}H_x(n)[f_z(n)T_x(n) - \mathcal{B}_x(n)]\mathrm{e}(hn/k) = O_A(L^{-A}). \tag{4.10}$$

Proof. Discard $n_y > D_0$ using the preceding bounds. In the genuine tuple term write $n = \ell a v$, where $a = n_y$ and v is y-rough. The coefficient is $z^d f_z(a)H_x(a)$, and $R = Y/(\ell a) \ge x^{\varepsilon-o(1)}$. Lemma 4.2 gives the principal term

$$\frac{z^d f_z(a)H_x(a)}{\ell a}\frac{c_k(ha\ell)}{\varphi(k)}\int \psi(u)u^{it}b_{z,y,N}(Yu/(\ell a))\,\mathrm{d}u, \tag{4.11}$$

where c_k is the Ramanujan sum over unit residues. For the model, write $n = av$ and use unweighted rough counting. Its V_y cancels the $1/V_y$ in (4.4). Since ℓ is a unit modulo k, $c_k(ha\ell) = c_k(ha)$, also when a is a nonunit. The model term is therefore exactly (4.11). The cutoff χ is one on these residual scales. Sum the errors with $\sum_a 1/a \le L^C$ and $\sum_{\boldsymbol{\ell}} 1/\ell = O(1)$, choosing precision after all test and time exponents. No regularity of $H_x(a)$ is needed. □

4.5. **The discrete high-time estimate.** Take the first comparison core $f_zT_x - \mathcal{B}_x$. Lemma 4.3 handles every fixed logarithmic range of low times. At high times, extract one small mark. For the genuine tuple contribution, extract one long prime as well. The identity $f_z(p_s p_\ell v) = z^2 f_z(v)$ leaves the two prime polynomials unchanged up to phases. Their residual coefficient has unchanged square norm, so the proof of Lemma 3.1 applies to this part.

For the model contribution, exact covariance gives $\mathcal{B}_x(p_s n) = z\mathcal{B}_x(n)$. At the remaining scale $R_0 \asymp Y/P_s$, write $n_y = a \le D_0$ and expand the roughness of n/a. The resulting terms are

$$\frac{z^d f_z(a)H_{\mathrm{rem}}(a)\mu(e)}{V_y\ell}\mathbf{1}_{ae\mid n}\,\beta\left(\frac{n}{a\ell}\right), \tag{4.12}$$

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where e is squarefree, $P^+(e) \le y$, and $\omega(e) \le h_0$. After writing $n = aev$, the normalized reciprocal coefficient mass is L^{C_*}. The smooth amplitude is $\beta(ev/\ell)$; its total variation obeys the same bound, with exponent independent of fixed N. All residual scales $R' = R_0/(ae)$ satisfy $R' \ge x^{1-o(1)}$.

The omitted coefficient has square mass $O_A(R_0 L^{-A})$. Ordinary Dirichlet mean square, applied before restriction to exceptional times, gives

$$R_0^{-2} \int_{|t|\le 2T_0} \left| \sum_n E_n n^{it} \right|^2 \mathrm{d}t \ll L^{-A}(T_0/R_0 + \log R_0), \qquad T_0/R_0 \asymp LP_s/H = o(1). \tag{4.13}$$

The last relation uses both lower bounds on U_i in (2.6), not only $Y = x^{1+o(1)}$.

Let $T_{\mathrm{mod}} = \exp(L/(\log L)^2)$. For $L^B \le |t| \le T_{\mathrm{mod}}$, separate the additive phase into residue classes modulo k. Sum–integral comparison and partial summation on each progression give a normalized bound

$$|N_0(t)| \ll L^{C_*}(|t|^{-1} + x^{-c}).$$

Indeed the discrete error is $O(L^C(1+|t|)/R')$, while integration by parts gives the $|t|^{-1}$ term. Consequently

$$\int_{L^B \le |t| \le T_{\mathrm{mod}}} |N_0(t)|^2 \,\mathrm{d}t \ll L^{2C_* - B} + x^{-c'}. \tag{4.14}$$

Logarithmic twists are absorbed by increasing B. For $T_{\mathrm{mod}} < |t| \le 2T_0$, the exact ranges in (2.10) apply at R' and give

$$|N_0(t)| \ll L^{C_*} \exp(-L/(\log L)^{C_{19}}). \tag{4.15}$$

The residue may be a nonunit; there is no inversion of ae.

Outside the exceptional small-prime intervals, the small polynomial and ordinary mean square on scale $Y/P_s \gg T_0$ suffice. On the exceptional intervals, integrate (4.14) directly and use (4.15) over at most $\exp(O(L^{\cdot 9}))$ unit intervals. The latter loss is absorbed by its exponential saving. This proves (2.7) for the model part. Choose B after the fixed kernel, group, and derivative exponents, then choose N after B and the low-time accuracy. The derivative exponents in Lemma 4.1 make this order possible.

Proposition 4.4 (Marked model transfer)**.** *For the data of Section 2, every fixed M, r and every required accuracy A, all sufficiently large fixed N satisfy*

$$\sum_{w \equiv r \ (M)} \frac{\Psi(w/x)}{w} \mathcal{W}(w)[f_z(w)T_x(w) - \mathcal{B}_x(w)]G(w+1)W_1(w)W_1(w+1) \ll_A L^{-A}. \tag{4.16}$$

One N can be used for every active set in a fixed extraction.

Proof. The first physical core is eventually bounded, so (2.4) gives a norm exponent independent of J, N. It remains list-independent. Use the original ideal family. On stripping shared labels, both terms have the exact covariance

$$[f_z T_x - \mathcal{B}_x](D_R w) = z^{Js_P - c_\nu}[f_z T_x - \mathcal{B}_x](w).$$

Divide out z^{Js_P}, restore the dyads and Mellin integral, and apply (3.1). The residual contains the raw difference and all comparison differences with factors z^{-c_ν}. The local and high-time estimates just proved bound every comparison, including fixed residue restrictions through Fourier shifts. Choose their accuracy after the total logarithmic pattern cost and subtract them. The same $\mathcal{B}_x$ and a sufficiently large single N work for every active set, because their exponents have a common fixed bound in each extraction. □

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5. Progression remainders and extraction of primes

5.1. **The complex progression remainder.** Retain fixed M, b with $(b, M) = 1$, $0 < c_0 < \varepsilon/8$, and a smooth nonnegative compact cutoff Φ. We first prove the estimate needed by the positive sieve.

Proposition 5.1. *For the inert-array weights H allowed in* (3.6)*, every $A > 0$, and sufficiently large fixed N,*

$$\sum_{\substack{r\le x^{c_0}\\(r,M)=1}}\left|\sum_{\substack{r\mid u+1\\u+1\equiv b\ (M)}}\Phi(u/x)H(u)[f_z(u)T_x(u)-\mathcal{B}_x(u)]\right|\ll_A xL^{-A}. \tag{5.1}$$

Proof. Here $H(u) = H(u_y)$ and is bounded for each fixed array collection. Discard $u_y > D_0$ by the positive divisor envelope (4.6) and the summed rounding estimate (4.8). In a genuine tuple term write $u = \ell av$, where $a = u_y \le D_0$ and $P^-(v) > y$. The long product ℓ is a unit modulo rM for large x. Set $g = (b-1, M)$. Compatibility requires

$$(a, r) = 1, \qquad (a, M) = g. \tag{5.2}$$

When these conditions hold, v lies in a reduced class modulo $q = rM/g$. Prime by prime, partial removal of a prime-power factor of M fixes the valuation of a; full removal allows larger valuations. In each case a/g is a unit modulo M/g. Roughness still excludes every prime at most y, including those fully removed from the modulus. The model term $u = av$ has the same compatibility and modulus, although its reduced residue can differ.

Insert (4.7) in the genuine progression sum and in its centered natural mean. Only divisors e coprime to q can contribute in a reduced class. Their residual scales satisfy $R/e \ge x^{\varepsilon-o(1)}$, and the distinct moduli rM/g are less than $(R/e)^{1/3}$. Apply (4.2), then sum with the $L^{O(1)}$ reciprocal divisor mass. The positive envelope controls progression errors by CRT; natural-mean errors cost only a logarithmic reciprocal-totient sum.

In the remaining natural means, the exclusion of primes $p \mid q$ with $p > y$ can be removed. After division by $\varphi(q)$, its total cost is bounded using

$$\sum_{q\le Mx^{c_0}}\frac{1}{\varphi(q)}\sum_{\substack{p\mid q\\p>y}}\frac{1}{p}\le\sum_{p>y}\frac{1}{p(p-1)}\sum_{t\le Mx^{c_0}/p}\frac{1}{\varphi(t)}\ll L^{O(1)}/y. \tag{5.3}$$

We used $\varphi(pt) \ge (p-1)\varphi(t)$, including $p \mid t$. This saves every logarithmic power. Lemma 4.2 at modulus one now gives, for fixed a, ℓ, principal contribution

$$\frac{xz^d f_z(a)H(a)}{a\ell\varphi(q)}\int\Phi(t)b_{z,y,N}(xt/(a\ell))\,\mathrm{d}t. \tag{5.4}$$

For the model, use unweighted rough counting in the reduced class of v. Its density is

$$\frac{V_y}{q\prod_{p\mid q,p\le y}(1-1/p)}=\frac{V_y}{\varphi(q)}\prod_{p\mid q,p>y}(1-1/p). \tag{5.5}$$

The $1/V_y$ in (4.4) cancels V_y. Equation (5.3) permits replacement of the final product by one. The result is exactly (5.4), including every retained density coefficient. These terms cancel before absolute values are taken. Sum the errors with the reciprocal a-mass $L^{O(1)}$ and tuple mass $O(1)$, using (4.8) for the integer-counting errors. □

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5.2. **A common positive sieve argument.** We now treat three first-endpoint differences:

$$D_x(u) = \begin{cases} T_x(u) - C_x, & \text{unsigned progression law,} \\ \lambda(u)T_x(u), & \text{parity,} \\ f_z(u)T_x(u) - \mathcal{B}_x(u), & \text{nonreal mode.} \end{cases} \tag{5.6}$$

Their summed progression remainders vanish to arbitrary logarithmic precision by (3.6), (3.7), and Proposition 5.1. For the unsigned case, subtract C_x times the remainder without the tuple.

Choose a large fixed even h, put $\gamma = c_0/(4h+3)$, and sieve $u+1$ by primes at most x^γ that do not divide M. For the unsigned case apply the block sieve separately to $\Phi\mathcal{W}T_x$ and $C_x\Phi\mathcal{W}$. For parity use the nonnegative weights $\Phi\mathcal{W}T_x(1\pm\lambda)/2$. For the complex case, $|\mathcal{B}_x| \le 1$ eventually, so with $A_0 = \Phi\mathcal{W}(T_x+1)$ the four weights

$$A_0 \pm \Phi\mathcal{W}\operatorname{Re} D_x, \qquad A_0 \pm \Phi\mathcal{W}\operatorname{Im} D_x$$

are nonnegative. All these weights include the same progression indicator. Each plus/minus pair has identical main terms and arbitrary-precision summed remainders. The block sieve is applicable because $x^{\gamma(4h+2)} < x^{c_0}$. Also

$$\frac{1}{M}\prod_{p\le x^\gamma, p\nmid M}(1-1/p) = \frac{1}{\varphi(M)}\prod_{p\le x^\gamma}(1-1/p)$$

for large x. Subtracting the paired estimates gives

$$\limsup_{x\to\infty}\frac{L}{xm_{\mathcal{W}}}\left|\sum_{\substack{u+1\equiv b\ (M)\\ P^-(u+1)>x^\gamma}}\Phi(u/x)\mathcal{W}(u)D_x(u)\right| \ll_{M,d,\varepsilon,\Phi} e^{-h}/\gamma. \tag{5.7}$$

The surviving constant is independent of K, h and the particular fixed inert array. Array-dependent remainder constants vanish in the first limit. In the complex case, the unsigned upper sieve and (4.5) bound the presieved model by $o(xm_{\mathcal{W}}/L)$ for fixed h and array. Thus (5.7) also holds with D_x replaced by f_zT_x.

5.3. **Replacing composite factors and retaining the coin coefficients.** A presieved composite in the support has a fixed bounded number j of prime factors, with $2 \le j \le 2/\gamma$ for large x. Repeated large factors have negligible absolute mass. Telescope each prime factor to the positive rough proxy of Section 2. The scalar criterion and (3.8) permit this for the unsigned, parity, and complex statistics. Fixed characters impose the product residue. Clipping, coefficient truncation, dyadic splitting, and logarithmic separation have the source's original absolute error bounds. Choose K after these fixed losses and γ, independently of the later inert exponents.

Write $v = \prod_{i=1}^j m_i$, $u = v-1$, and let $\mathcal{C}$ be the full candidate set of small and big groups. A bar on m_i removes all candidate prime parts, including inactive groups. The stripped coefficient is

$$\widetilde{a}(m_i) = \mathbf{1}_{P^-(m_i)>W}\mathbf{1}_{\bar{m}_i>x^\gamma}\kappa(\bar{m}_i), \qquad \kappa(t) = \frac{1}{V(W)\log t}.$$

The source bounds its total positive tuple mass by $O_{j,\gamma}(x/L)$ and the threshold/logarithm replacement error by $o(x/L)$.

Conditional on the tuple, the source uses independent coins with probabilities

$$s_g(u,v) = c_1 q_0^{\omega_g(u)-1}\frac{\omega_g(u)}{V_g}(q_0/j)^{\omega_g(v)-1}\frac{\omega_g(v)}{V_g}. \tag{5.8}$$

Choose $c_1 > 0$ so that these numbers lie in $[0,1]$; a zero hit count gives zero. Let $\mathcal{R}$ be the success sets with at least $c_2\log L$ groups of each kind. The source's success-probability estimate says that

the other outcomes have total positive proxy mass $o(x/L)$. Discard them at this stage, before expanding signs. All first-endpoint statistics under consideration are bounded.

The retained-event probability has the exact expansion

$$\begin{aligned} R(u,v) &= \sum_{S\in\mathcal{R}} \prod_{g\in S} s_g \prod_{g\notin S}(1-s_g) \\ &= \sum_{S\in\mathcal{R}} \sum_{T\subseteq\mathcal{C}\setminus S} (-1)^{|T|} \prod_{g\in S\cup T} s_g, \qquad 0\le R(u,v)\le 1. \end{aligned} \tag{5.9}$$

Each active set $P = S\cup T$ meets the hypotheses of the marked correlation, and the number and coefficient mass of terms are $L^{O(1)}$.

Outside candidate-prime-square exceptions, strip the active part of v, obtaining v_*. For each factorization $e_1\cdots e_j = v_*$, the active primes can be allocated independently to the j slots. The bars make the coefficients identical for all allocations. The identity

$$j^{\omega_P(v)}(q_0/j)^{\omega_P(v)-|P|} = j^{|P|}q_0^{\omega_P(v)-|P|} \tag{5.10}$$

then gives the invariant core

$$G_P(v) = \sum_{e_1\cdots e_j=v_*} \prod_{i=1}^{j} \left[\mathbf{1}_{P^-(e_i)>W}\mathbf{1}_{\bar{e}_i>x^\gamma}\kappa(\bar{e}_i)\right].$$

It satisfies $G_P(v) = G_P(v_*)$ and $|G_P(v)| \le L^{C_j}\tau(v_*)^{C_j}$. Every expanded term has its actual coefficient $(-1)^{|T|}c_1^{|P|}j^{|P|}$ multiplying the marked sum. Candidate-square exceptions have exponentially small divisor-weighted mass and remain negligible after this logarithmic expansion cost.

For the unsigned and parity cases, apply (3.2) or Lemma 3.1, with $r = b-1$ and $\Psi(t) = t\Phi(t)$, and multiply by x. Choose the accuracy after the expansion costs. This makes the composite contribution $o(x/L)$.

For a nonreal mode, Proposition 4.4 replaces the first core by $\mathcal{B}_x(u)$ with arbitrary logarithmic precision. The principal contribution for a given P is

$$\sum_{u+1\equiv b\ (M)} \Phi(u/x)\mathcal{W}(u)\mathcal{B}_x(u)W_{1,P}(u)W_{1,P}(u+1)G_P(u+1).$$

Sum these terms with all their coefficients before estimating their absolute values. Reversing (5.10) and then (5.9) gives exactly

$$\sum_{m_1,\dots,m_j} \mathbf{1}_{v\equiv b\ (M)}\Phi((v-1)/x)\mathcal{W}(v-1)\mathcal{B}_x(v-1)R(v-1,v)\prod_i \widetilde{a}(m_i), \tag{5.11}$$

up to the already controlled square exceptions. The same $\mathcal{B}_x$ occurs for every active set, so it factors through both identities. All other factors in (5.11) are nonnegative, and $R\le 1$. Hence its absolute value is at most

$$\sup_{u\asymp x}|\mathcal{B}_x(u)|\, O_{j,K,\gamma,\Phi}(x/L) = o(x/L). \tag{5.12}$$

The source mass bound follows from $O(xV(W)^j)$ rough tuples per product box, coefficient size $O_\gamma((LV(W))^{-j})$, and $O(L^{j-1})$ boxes. The fixed inert weight costs a fixed constant. Logarithmic expansion factors multiply only the arbitrary-precision errors, not (5.12). This distinction permits roots z whose fixed positive exponent in (4.5) is small.

5.4. **Removing the inert marks.** We have bounded the prime sum with each fixed inert array by the right side of (5.7). For any fixed integer $B\ge 1$, choose B disjoint arrays of the same sufficient size K, and put

$$Z_B(u) = \left(\frac{1}{B}\sum_{a=1}^{B}\frac{\mathcal{W}_a(u)}{m_a} - 1\right)^2.$$

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The source nonnegative deweighting estimate is

$$\limsup_{x\to\infty}\frac{L}{x}\sum_{p}\Phi((p-1)/x)Z_B(p-1)\ll C_K/B. \tag{5.13}$$

It applies without a progression restriction. Restriction decreases its left side. The other Cauchy factor is $O((x/L)^{1/2})$, because T_x-C_x, λT_x, and f_zT_x are bounded on the support. Consequently the unweighted statistics satisfy

$$\limsup_{x\to\infty}\frac{L}{x}\left|\sum_{p\equiv b\ (M)}\Phi((p-1)/x)F_x(p-1)\right|\le C_{M,d,\varepsilon,\Phi}\frac{e^{-h}}{\gamma}+C_{d,\varepsilon,\Phi}\sqrt{C_K/B}, \tag{5.14}$$

where F_x is respectively T_x-C_x, λT_x, or f_zT_x. The model term has already been removed in the last case.

The order of limits is part of the proof. Fix the progression and interior data, then h,γ, the finite range of j, and a sufficient K. Fix any finite number B of arrays. Choose the physical families, high-time exponents, density order N, and error precisions. Take $x\to\infty$ first, then $B\to\infty$, and finally $h\to\infty$, choosing a new sufficient fixed K for each h. Since $\gamma^{-1}=(4h+3)/c_0$, (5.14) tends to zero. Constants depending on N,K,B,γ in errors have disappeared in the first limit.

Theorem 5.2 (Interior estimates)**.** *Under Assumption 1.1, for every fixed reduced class* $b \pmod M$*, every nonempty interior tuple statistic* (2.1)*, and every* $\Phi\in C_c^\infty((0,\infty))$*,*

$$\sum_{p\equiv b\ (M)}\Phi((p-1)/x)(T_x(p-1)-C_x)=o(x/L), \tag{5.15}$$

$$\sum_{p\equiv b\ (M)}\Phi((p-1)/x)z^{\Omega(p-1)}T_x(p-1)=o(x/L)\quad(z\neq 1), \tag{5.16}$$

where z *is any fixed root of unity different from one.*

The proof used $\Phi\ge 0$ for the positive sieve. A real smooth compact cutoff is a difference of two such cutoffs, obtained with a smooth dominating bump. Real and imaginary parts then give the complex formulation by linearity.

6. Probability limits and squarefree restrictions

6.1. The unsigned law in each progression.

Proposition 6.1. *Under Assumption 1.1,* $V(p)$ *converges in each fixed finite set of coordinates to* PD(1) *when* p *ranges over a fixed reduced progression, in either the dyadic or cumulative convention.*

Proof. First take p uniform in $(x,2x]$ in the class. The denominator $N_M(x)$ is asymptotic to $x/(\varphi(M)L)$ by the fixed-modulus prime number theorem. For rectangles $\prod_{i=1}^d(a_i,b_i]$ with positive lower endpoints and $\sum b_i<1$, choose prime slots $(x^{a_i},x^{b_i}]$. Their reciprocal tuple sum tends to $\prod_i\log(b_i/a_i)$; coincidences contribute $o(1)$. Equation (5.15) and smooth approximation of the dyadic cutoff give the same limit for the mean tuple count.

Regard the prime factors of $p-1$, with multiplicity, as labelled balls of masses $\log q/\log(p-1)$. On a compact subset of

$$\mathcal{D}_d=\{(t_1,\dots,t_d):t_i>0,\ \sum_i t_i<1\},$$

the expected ordered distinct-ball measure therefore converges to $\prod_i \mathrm{d}t_i/t_i$. The change from $\log x$ to $\log(p-1)$ is $1+O(1/L)$ and is controlled by endpoint strips. Repeated numerical primes on a fixed interior compact set entail $q^2\mid p-1$ with $q\ge x^{a/2}$ for some fixed $a>0$. Their integer count is $O(x^{1-a/2}+\sqrt{x})=o(N_M(x))$. Tuple counts are bounded on such a compact set, so multiplicities cause no further error.

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Sample balls without replacement with probability proportional to their remaining masses. The first d size-biased draws multiply the factorial density by

$$\prod_{i=1}^{d} \frac{t_i}{1-\sum_{j<i} t_j}.$$

The resulting density is $h_d(t) = \prod_{i=1}^{d}(1-\sum_{j<i} t_j)^{-1}$ on $\mathcal{D}_d$. The triangular substitution $u_i = t_i/(1-\sum_{j<i} t_j)$ identifies it with uniform measure on $(0,1)^d$. Its mass is one. Compact exhaustion therefore excludes escaped boundary mass and proves convergence of the size-biased draws to the first d GEM(1) fragments.

Let $R_{x,d}$ be the unseen mass after d draws, with zero padding after exhaustion. Its expectation tends to 2^{-d}. Sorting the first d draws and padding by zero approximates every ranked coordinate of $V(p)$ within $R_{x,d}$. Uniform continuity and Markov's inequality, followed by $d \to \infty$, prove the ranked finite-dimensional law. A finite decomposition of $(X/2^J, X]$ into dyads gives the cumulative law. The omitted initial class-prime proportion tends to 2^{-J}; take $X \to \infty$ and then $J \to \infty$. □

6.2. **Complex measures and Fourier inversion.** Attach $f_z(p-1)$ to each sampled prime, for a fixed $z \neq 1$. The resulting complex factorial measure $\sigma^z_{x,d}$ has total variation dominated by its unsigned counterpart. Equation (5.16) implies

$$\int \phi \, \mathrm{d}\sigma^z_{x,d} \longrightarrow 0, \qquad \phi \in C_c(\mathcal{D}_d).$$

The same unsigned endpoint-strip and multiplicity bounds used above justify the normalization and rectangle-to-continuous-test passage in absolute value.

Let $\nu_{x,d}$ be the unsigned size-biased law and $\nu^z_{x,d}$ its complex weighted version. On the interior,

$$\mathrm{d}\nu^z_{x,d} = \prod_i \frac{t_i}{1-\sum_{j<i} t_j} \, \mathrm{d}\sigma^z_{x,d}, \qquad |\nu^z_{x,d}| \le \nu_{x,d}.$$

Choose $0 \le \chi \le 1$ compactly supported in $\mathcal{D}_d$ with $\int \chi h_d > 1-\eta$. For any continuous H on the cube,

$$\left| \int H \, \mathrm{d}\nu^z_{x,d} \right| \le \left| \int \chi H \, \mathrm{d}\nu^z_{x,d} \right| + \|H\|_\infty \int (1-\chi) \, \mathrm{d}\nu_{x,d}.$$

The first term tends to zero, and the limsup of the second is at most $\eta\|H\|_\infty$. Let $\eta \to 0$. Thus $\nu^z_{x,d}$ converges weakly to the zero complex measure on the whole cube. This is weak convergence, not convergence in total variation. At $d = 1$ its total mass is exactly $\mathbb{E} f_z(p-1)$, which proves cancellation without a separate $d = 0$ correlation.

If H tests k ranked coordinates, the error from sorting d draws, with $d \ge k$, is at most

$$\omega_H(\eta) + 2\|H\|_\infty \mathbb{E} R_{x,d}/\eta.$$

The same bound holds for the complex weighted average because $|f_z| = 1$. Take $x \to \infty$, then $d \to \infty$, then $\eta \to 0$. We obtain

$$\mathbb{E}[f_z(p-1)H(V_1(p), \dots, V_k(p))] \longrightarrow 0 \quad (z \neq 1). \tag{6.1}$$

For fixed m, finite Fourier inversion gives

$$\mathbf{1}_{\Omega(n) \equiv a \ (m)} = \frac{1}{m} \sum_{j=0}^{m-1} \exp(-2\pi i j a/m) \exp(2\pi i j \Omega(n)/m).$$

The trivial mode is Proposition 6.1; $z = -1$ is the parity case; the other modes were treated in Section 4. This proves Theorem 1.2 in dyads. The same finite dyadic decomposition proves its cumulative form.

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6.3. **Local square exclusions.** Let $B_Y(n)$ indicate that $q^2 \nmid n$ for every prime $q \le Y$. Within a fixed reduced class modulo M, this condition on $p-1$ is a finite union of reduced classes modulo $\operatorname{lcm}(M, \prod_{q\le Y} q^2)$, apart from finitely many primes. For fixed Y, apply the unsigned law and (6.1) with $z=-1$ in each class. Reduced-lift counting gives the factors $1-\rho_q(M,b)$ in (1.2). If $q \nmid M$, there are $q(q-1)$ reduced lifts modulo q^2 and one has $p \equiv 1 \pmod{q^2}$. If $v_q(M)=1$ and $b \equiv 1 \pmod q$, one of q lifts has that property. If $q^2 \mid M$, it is forced or excluded by b itself.

To let Y grow after x, use

$$\limsup_{x\to\infty} \frac{\#\{x < p \le 2x : \exists q > Y, q^2 \mid p-1\}}{\pi(2x)-\pi(x)} \ll 1/Y. \tag{6.2}$$

For $Y < q \le x^{1/4}$, Brun–Titchmarsh (in the form recorded in [7]) gives $O(x/[q(q-1)\log x])$ primes in the relevant progression. For $q > x^{1/4}$, count integers and use

$$\sum_{x^{1/4}<q\le\sqrt{2x}} (x/q^2+1) = O(x^{3/4}+\sqrt{x}) = o(x/\log x).$$

The first reciprocal tail is $O(1/Y)$. Changing the denominator in (6.2) to $N_M(x)$ costs only a fixed factor. Bounded tests and signs preserve the bound.

Write $B=\mu^2$ temporarily for the arithmetic indicator. Taking $x\to\infty$ and then $Y\to\infty$ yields the four limits

$$\begin{aligned} \mathbb{E}H(V(p)) &\longrightarrow \mathbb{E}H(L), & \mathbb{E}[\lambda(p-1)H(V(p))] &\longrightarrow 0,\\ \mathbb{E}[B(p-1)H(V(p))] &\longrightarrow C_{M,b}\mathbb{E}H(L), & \mathbb{E}[\lambda(p-1)B(p-1)H(V(p))] &\longrightarrow 0. \end{aligned}$$

The identities

$$\mathbf{1}_{\lambda=s,B=1} = B(1+s\lambda)/2, \qquad \mathbf{1}_{\lambda=s,B=0} = (1-B)(1+s\lambda)/2$$

give every joint atom. This proves Theorem 1.3, including $C_{M,b}=0$. Finally $\mu=\lambda\mu^2$ gives (1.3) and the two Möbius-sign densities.

6.4. **The prescribed-valuation consequence.** On the event $v_q(p-1)=e_q$ for $q\in\mathcal{S}$, let $B_{\mathcal{S}}$ indicate squarefreeness outside $\mathcal{S}$. Then

$$\mu_{\mathcal{S}}(p-1) = (-1)^{\sum_{q\in\mathcal{S}} e_q}\lambda(p-1)B_{\mathcal{S}}(p-1).$$

For fixed Y, the valuation event and the square exclusions at $q\le Y$, $q\notin\mathcal{S}$, are a finite union of reduced classes modulo

$$\operatorname{lcm}\left(\prod_{q\in\mathcal{S}} q^{e_q+1}, \prod_{\substack{q\le Y\\ q\notin\mathcal{S}}} q^2\right).$$

Apply (6.1) with $z=-1$ and $H=1$ in each class. Equation (6.2) controls the remaining square factors. Taking the limits in that order proves Corollary 1.4. The argument permits $e_q\ge 2$, because squarefreeness is imposed only outside $\mathcal{S}$.

Appendix A. Analytic dependencies

The results are conditional on the following inputs, with the hypotheses and normalizations stated in Section 2. References [1, 2, 3, 4] refer throughout to the same fixed repository revision.[1] Proof estimates mentioned below are included among the assumptions.

[1]Revision `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb` of `openai/math`. The bibliography links to these versions.

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Transference and marked correlations. We use the general graph-transference and residual ideal-family propositions in [1, Section 6], which apply the local transference, endpoint pairing, and ideal-kernel results of [2, Sections 3–4]. The required endpoint norms, list normalization, shared-label collision bound, and bilinear multiplier estimate are those used in their proofs. The local Mellin-energy lemma, truncated divisor-model lemma, and sparse Gram bound in [1, Section 6] give the reduction (2.7) and expansion (2.8). In particular, the norm exponent is independent of the fixed shared-list parameter J. Sections 3 and 4 verify the new endpoint energy estimates; no signed or fixed-progression version of the final distribution theorem is assumed.

Oscillation and the scalar estimate. The long-prime polynomial and logarithmic-phase lemmas in [1, Appendix A] are used in the ranges (2.9)–(2.10), including nonunit residue classes in the latter. For the determinant replacement, we use the coefficient criterion for the marked estimate in [3, Section 8] and the scalar Cauchy-square bound in its proof. Its coefficient hypothesis is (2.11); the prime-minus-rough coefficient satisfies it by the minor-moment and major-term estimates in [1, Sections 3–5]. Equation (3.8) keeps the new multiplicative factor outside that scalar square.

Positive extraction. We use the block-sieve and divisor-moment estimates in [1, Section 2], followed by the progression, prime-to-rough, conditional-coin, allocation, and mean-square deweighting estimates in [1, Section 7]. These include the proof estimates for removing candidate prime parts from the rough coefficients and thresholds. The sieve level, positive proxy mass, and deweighting bound used here are stated in Sections 2 and 5. The complex progression remainder and the recombination of its principal terms are proved in Section 5.

Ordinary multiplicative-function estimates. The main theorem of [4] supplies the zero-free half-plane for Dirichlet L-functions required in Section 4. We use this assertion as a hypothesis. The remaining distribution inputs are the Liouville estimate recorded in [11, equation (1.5)], Granville–Shao's criterion [8, Theorem 1.1(b)], and the Selberg–Delange expansion [5, Theorem 1.2 and equation (1.10)], as applied in Sections 3 and 4.

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