A twelve-vertex counterexample to the 3-extendability conjecture for 8-regular optimal 1-embedded graphs on the Klein bottle, with a kernel-checked literal Lean endpoint

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Target: MathDB 366521, "3-extendability conjecture for optimal 1-embedded graphs on the Klein bottle"; source: S. Koizumi and Y. Suzuki, *Connectivity and matching extendability of optimal 1-embedded graphs on the torus*, arXiv:2501.02726v2 (7 Jan 2025), Conjecture 25 (printed p. 13). Version PAPER-v1, 2026-11-15.

1. The statement

Source definitions (pp. 1-2, read in the owned v2 text and in rendered pages 1 and 13). All graphs are finite, simple and connected. A graph is 1-embeddable on a closed surface F if it can be drawn on F so that every edge crosses another edge at most once; drawings are good: (i) vertices are distinct points, (ii) adjacent edges do not cross, (iii) no three edges cross at one point, (iv) nonadjacent edges do not touch tangentially. A 1-embedded graph on F with at least three vertices has |E| <= 4|V| - 4chi(F); it is optimal when equality holds. A matching M is extendable if some perfect matching contains M; a graph with at least 2m+2 vertices is m-extendable if every matching of size m is extendable.

Conjecture 25: every 8-regular optimal 1-embedded graph on the Klein bottle is 3-extendable. Since chi(Klein bottle) = 0, optimal means |E| = 4|V|.

Theorem. There is a finite simple connected 8-regular graph G on 12 vertices with 48 = 4*12 edges, a good drawing of G on the Klein bottle in which every edge has at most one crossing point, and a matching M of 3 edges contained in no perfect matching of G. Hence Conjecture 25 is false. The order 12 is even and at least 8, so this is not an odd-order or small-order artefact.

2. The surface

Let R = [0,4] x [0,3]. Define the boundary normal form nf: corners go to (0,0); (4,y) goes to (0,y); (x,3) goes to (4-x,0); every other point is fixed. Put p ~ q iff nf p = nf q, and K = R/~ with the quotient topology. The classes are: single interior points, pairs {(0,y),(4,y)} for 0<y<3, pairs {(x,0),(4-x,3)} for 0<x<4, and the four corners. Gluing theorem (Lean mk_eq_iff_eqvGen): ~ is exactly the equivalence relation generated by the two standard side pairings (0,y)~(4,y) and (x,0)~(4-x,3). Proof: each pairing preserves nf (case split on corners); conversely every point is joined to its normal form by at most two pairings, e.g. (0,3)~(4,3)~(0,0). So K is the standard reflected-rectangle Klein bottle, presented with no second model and no homeomorphism to another model.

K is compact (image of the compact R) and Hausdorff: the polynomial map F(x,y) = (x(4-x), y(3-y), x(4-x)(x-2)(3-2y), y(3-y)(2y-3), x(4-x)(x-2)y(3-y)) is constant on classes and separates normal forms (the positivity pattern of the first two coordinates gives the type of normal form; inside a type one cancels a positive factor from the third, fourth or fifth coordinate), so it induces a continuous injection K -> R^5.

Interior chart. With U = (0,4)x(0,3), the map iota: U -> K is an open embedding: nf is the identity on U and the only point of R with an interior normal form is that point, so iota is injective and the full preimage of iota(V) is V itself for V open in U.

Seam charts. On S_a = (0,4)x(-1,1), Phi_a(x,y) = [x,y] for y >= 0 and [4-x,3+y] for y < 0 is an open embedding: it is continuous because the two formulas agree on y = 0 ((x,0)~(4-x,3)); it is injective by comparing normal forms; and its

mathdb:2611.00110v1 17 Nov 2026

images of open sets have explicit open saturated preimages {(a,b): b<1, (a,b) in O} union {(a,b): b>2, (4-a,b-3) in O}. Symmetrically Phi_b(x,y) = [x,y] for x >= 0 and [4+x,y] for x < 0 on S_b = (-1,1)x(0,3).

3. The graph

Vertices are 0,...,11, with v = 4y + x for the lattice point (x,y), 0<=x<=3, 0<=y<=2. The twelve unit squares of R give, after identification, the faces
[0,1,5,4] [1,2,6,5] [2,3,7,6] [3,0,4,7] [4,5,9,8] [5,6,10,9] [6,7,11,10] [7,4,8,11] [8,9,3,0] [9,10,2,3] [10,11,1,2] [11,8,0,1]. G joins two distinct vertices iff they lie in a common face (each face is a 4-clique). Each vertex has two horizontal neighbours in its row and three neighbours in each of the other two rows, eight in all; so G is 8-regular and |E| = 12*8/2 = 48 (Lean: degree count by kernel evaluation, edge count by the handshake lemma). Every vertex is within distance two of vertex 0, so G is connected.

4. The drawing

Vertex v is drawn at [x_v, y_v]. These points are pairwise distinct because nf fixes the lattice representatives. The 48 edges are drawn as the quotient images of the 48 affine unit segments of the squares [i,i+1]x[j,j+1] (i<4, j<3): horizontal sides (i+t, j), vertical sides (i, j+t), diagonals (i+t, j+t) and (i+1-t, j+t), t in [0,1]. Each lies in R, its endpoints are the lattice points whose labels give the edge (seam endpoints such as (4,j) or (i,3) are identified with their labelled vertex by nf), and every edge of G arises from exactly one segment.

Key observation: at every OPEN segment parameter 0<t<1, nf fixes the point (horizontal sides have 0<x<4 and y in {0,1,2}; vertical sides have x in {0,1,2,3} and 0<y<3; diagonals lie in U). Consequently every coincidence of open edge points in K is a coincidence in the plane. Then:
- Edge interiors avoid vertices: an open segment point has a non-integer coordinate.
- Each arc is injective: open points by planar injectivity, endpoints by distinct labels and vertex injectivity.
- Intersection classification (seg_meet, arc_meet): open points of two segments with indices k != l coincide only if k, l are the two diagonals of one square, and then only at the centre (t = 1/2 on both). The proof is a 16-case analysis of linear equations with integer offsets, using that no integer lies strictly between 0 and 1. diagonals_cross shows the centre really is a crossing.
- The two diagonals of a square have four distinct endpoints (finite check), so adjacent edges never cross; no point lies on three edge interiors; each edge has at most one crossing point (sides have none).
- Transversality in a genuine chart: near the centre c of square (i,j) the chart iota exhibits the diagonals as c + (s-1/2)(1,1) and c + (t-1/2)(-1,1), with determinant 2 != 0. Thus the drawing is a good 1-embedding (Lean kleinDrawing_good proves the full predicate IsGoodOneDrawing).

5. Euler characteristic and optimality

K carries the canonical one-vertex cell structure of the rectangle model: 0-cell [0,0]; 1-cells a(s) = [4s,0] and b(s) = [0,3s]; 2-cell f(s,t) = [4s,3t]. Each characteristic map is continuous on the closed cube; its restriction to the open cube is an embedding (the 2-cell through the interior chart, the 1-cells through the seam charts Phi_a, Phi_b); cube boundaries map into lower cells; and the four open cells partition K (classification of normal forms). With K Hausdorff this is a finite CW complex, and chi = 1 - 2 + 1 = 0. Therefore 4|V| - 4chi = 48 = |E| and the 1-embedding is optimal. The Lean statement Conjecture25 K C takes chi from such a certified cell structure C; conjecture25_false_of_euler_zero refutes it for every cell structure with chi = 0. Disclosed limit: the classical topological invariance of the Euler characteristic (independence of the cell structure) is used to identify this number with the chi(F) of the source; that invariance theorem is not formalized.

6. The matching

M = {15, 39, 7 11}: each is a square side ({1,5} in [0,1,5,4], {3,9} in [8,9,3,0], {7,11} in [6,7,11,10]) and the six endpoints are the odd vertices. Removing them leaves {0,4,8} and {2,6,10}, each a triangle, with no edge between them. In a perfect matching P containing M every odd vertex is matched by its M-edge, so the partner of a vertex of {0,4,8} is even, hence in {0,4,8} (no even neighbour outside); a fixed-point-free partner involution on three vertices is

mathdb:2611.00110v1 17 Nov 2026

impossible. So M is in no perfect matching and G is not 3-extendable, although |V| = 12 >= 8. (Lean: matchM_not_extendable, using Mathlib's Subgraph.IsMatching and IsPerfectMatching.)

7. The Lean endpoint

Pinned Lean 4.34.1 and Mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612 (binaries checked by SHA256 in the frozen wrapper; specific Mathlib imports only). Seven new modules, in dependency order: KleinDefs (source predicates), KleinGraph, KleinSurface, KleinDrawing, KleinCells, KleinT2, KleinFinal. Main results:
- Klein366521.conjecture25_false : ¬ Conjecture25 KleinBottle kleinCells
- Klein366521.conjecture25_false_of_euler_zero
- Klein366521.klein_counterexample: T2 and compact surface, chi = 0, IsGoodOneDrawing kleinDrawing, connected, 8-regular, 48 edges, the optimality equation, 8 <= 12, M is a matching with edge set {s(1,5), s(3,9), s(7,11)} and 3 edges, no perfect matching contains M, not 3-extendable.
- Klein366521.kleinBottle_gluing, Klein366521.diagonals_cross.
#print axioms on all of these reports exactly [propext, Classical.choice, Quot.sound]. No sorry, admit, new axiom, unsafe, native_decide or set_option. Finite facts use decide and decide +kernel (kernel reduction, no extra axiom).

Source-to-Lean bridge:
- finite simple connected graph: SimpleGraph W, [Fintype W], H.Connected.
- 8-regular: H.IsRegularOfDegree 8.
- drawn on the surface: Drawing H K, a point per vertex and a continuous map from [0,1] per edge, running between its endpoints.
- good drawing, every edge crossed at most once: IsGoodOneDrawing = vertex injectivity, injective arcs, interiors avoid vertices, adjacent interiors disjoint, no triple interior point, every crossing transverse in an open-embedding chart from an open planar set, and at most one crossing point per edge.
- optimal: (|E| : ℤ) = 4|W| - 4 * C.euler for a certified cell structure C of K.
- 3-extendable: IsMExtendable H 3 = 8 <= |W| and every Mathlib matching with 3 edges lies in a Mathlib perfect matching.
- Klein bottle: KleinBottle = closed rectangle modulo exactly the standard side pairings (gluing theorem), compact and Hausdorff. The conjecture is quantified over all finite types with decidability instances; these only fix computational presentations and do not change the content.

8. Limits and contribution

The ordinary proof was found by a producer lane and independently passed by a reviewer, who rated the contribution BORDERLINE (modest target-specific counterexample; complete routine width-4/odd-height specialisation exists; no authenticated earlier public resolution found). This paper adds the literal surface/topology formalisation only. The final journal version (DOI 10.7151/dmgt.2609) and the projective-plane companion (DOI 10.7151/dmgt.2581) remain unread because of a publisher TLS failure; no certificate bypass or alternate host was used. No novelty, priority or significance claim is made here, and no credit.

Appendix: complete Lean modules, in dependency order

Each block below is the exact UTF-8 content of the compiled source file, with its SHA256.

src/KleinDefs.lean

SHA256 711bffdeaecf54f6e3d1033639eaf52d7a6de02861c1979268884239b1a60385, 5534 bytes, 104 lines.

Literal Lean source appendix: publication/lean-appendix-61.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinGraph.lean

SHA256 5d24c0b4c66317f163519981914c86ef1fe42947c3808a3a657d840da1910210, 7583 bytes, 180 lines.

mathdb:2611.00110v1 17 Nov 2026

Literal Lean source appendix: publication/lean-appendix-62.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinSurface.lean

SHA256 0db657ff08b971058e4f04ccf1f781c779325a1dc75fd584d88e9b3ccf57d8b8, 13166 bytes, 287 lines.

Literal Lean source appendix: publication/lean-appendix-63.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinDrawing.lean

SHA256 ec197556caf64f1b0164c8461328bfbdc7697a53c3535d5f398f96a2278df733, 22463 bytes, 511 lines.

Literal Lean source appendix: publication/lean-appendix-64.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinCells.lean

SHA256 be7e3b083e7edeba7c16dad8e4ea13c28bcf8f10717e51b026b31a029fb99fc0, 31572 bytes, 677 lines.

Literal Lean source appendix: publication/lean-appendix-65.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinT2.lean

SHA256 aacbc1fd53a66b18eb9afaf7244b2d2bd90d37d59ca1492b40f150c7c0150749, 5915 bytes, 141 lines.

Literal Lean source appendix: publication/lean-appendix-66.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

src/KleinFinal.lean

SHA256 6b6d18816b3080e1607e2c1b440c543544186ba9116b695c4eb567fde4021d70, 3722 bytes, 79 lines.

Literal Lean source appendix: publication/lean-appendix-67.md in the accompanying proof-sources.json / supporting.zip. All proof source bytes are supplied separately.

mathdb:2611.00110v1 17 Nov 2026