Universal symmetries of directed compass boards

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2026-11-15. Implementation certificate for MathDB 352193, source-interpreted as Ary Shaviv, *Board games, random boards and long boards*, arXiv:2110.05416v1 (11 October 2021), Definition 2, Definition 11, Section 5.1, Conjecture 15.

Fix an odd $n\geq 3$, write $n=2k+1$, $k\geq 1$, and $m=n-1$. Positions are pairs (i,j) with $0\leq i,j\leq m$, increasing south and east. Directions are N,NE,E,SE,S,SW,W,NW. A board assigns an arbitrary direction to every position. Its graph has every position as a vertex and every positive integral step along its assigned ray as a directed edge, including outgoing edges at the center. The distinguished roots are $s=(0,0)$ and $c=(k,k)$. Graph isomorphisms must fix each root. For arbitrary permutations σ of positions and τ of directions, put $S(A)(v)=\tau(A(\sigma(v)))$. The universal hypothesis is that for every A there exists a rooted graph isomorphism from A to S(A); the isomorphism may depend on A.

Theorem. Such S are exactly identity and physical main-diagonal reflection $R(A)(i,j)=r(A(j,i))$, where r swaps N/W, E/S, NE/SW and fixes SE,NW. In fact the representing pair (σ,τ) is respectively (Id,Id) or (transpose,r), and both possibilities occur.

Proof. Let $L_v(d)$ be the number of vertices strictly ahead along ray d, and $E(A)=\Sigma_v L_v(A(v))$. Every arc has exactly one source, so E is the complete arc count and is invariant under every graph isomorphism. Beginning-root outdegree is invariant as well. These invariants do not identify σ with the vertex map.

First separate directions without using diagonal sum formulas. All four cardinal totals $T(d)=\Sigma_v L_v(d)$ are equal. Indeed $L_N=i$, $L_S=m-i$, $L_W=j$, $L_E=m-j$; reversing the row interchanges N/S totals, and swapping coordinates interchanges N/W and E/S totals. For NE or NW, project the target of each diagonal ray to the same row on the source column. This injects its ray targets into those of N. The step length is determined by the row difference, so projection is injective. For SE or SW use S instead. At a corner the relevant cardinal ray is nonempty while the diagonal ray is empty: use bottom-right for NE, bottom-left for NW, top-right for SE, top-left for SW. Thus every diagonal total is strictly below the common cardinal total. This proof works for every $k\geq 1$, including $k=1$. A constant board d transforms to the constant board $\tau(d)$; consequently $T(d)=T(\tau(d))$. Strict separation implies that τ preserves the cardinal subset in both directions.

Now independently vary the input at $\sigma(v)$, between any two directions d,e. Bijectivity of σ means that only output v varies. Subtract the two invariant arc-count equations in the integers:
$L_v(\tau(d))-L_v(\tau(e))=L_{\sigma(v)}(d)-L_{\sigma(v)}(e)$. Hence there is an integer b_v independent of d with
$L_v(\tau(d))=L_{\sigma(v)}(d)+b_v$. Since $L_N+L_S=m=L_E+L_W$ everywhere, cancellation gives
$L_v(\tau(N))+L_v(\tau(S))=L_v(\tau(E))+L_v(\tau(W))$
at every v. If $\tau(N),\tau(S)$ were adjacent cardinals, take the corner where both point outwards. Their lengths are zero, and the two complementary cardinal lengths are m each, contradicting $m>0$. Thus opposite cardinal pairs are preserved.

The beginning-root invariant gives $L_s(A(s))=L_s(\tau(A(\sigma(s))))$ for every A. If $\sigma(s)\neq s$, vary A(s) between N and E while keeping $A(\sigma(s))$ fixed; the two sides cannot remain equal, since these lengths are 0 and m. Thus $\sigma(s)=s$. Constant-board root invariance gives $L_s(d)=L_s(\tau(d))$. The inward cardinal set {E,S} is therefore preserved, as is its outward complement {N,W}. Together with opposite preservation this gives exactly two cardinal actions: τ fixes N,S,W, or $\tau(N)=W$, $\tau(S)=E$, $\tau(W)=N$. The Lean proof closes this same finite compass calculation using cardinal preservation, injectivity, constant-board root equations and opposite balance at the upper-right corner. It does not enumerate board sizes.

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Physical reflection is a rooted graph isomorphism: transpose maps a positive ray step to the corresponding reflected ray step, is bijective, and fixes both s and c. Normalize the second cardinal case by composing R after S. With the pullback convention the normalized permutations are σ'=σ∘transpose and τ'=r∘τ. The normalized map remains universal and fixes N,S,W.

For this normalized map, add the local equations for N,S: m=m+2b_v, so b_v=0 in the integers for every vertex. The N and W equations now recover the row and column of σ(v), proving σ=Id. Thus L_v(τ(d))=L_v(d) everywhere. The eight ray-length profiles are distinct: at appropriate corners their zero/positive patterns separate every pair. Therefore τ=Id. Undoing reflection proves the two global alternatives. Identity and reflection have explicit rooted isomorphisms, proving sufficiency.

To pass to the source group of board permutations, the product action is a bijection: its inverse uses σ^{-1} and τ^{-1}. The representation is faithful. Constant boards recover τ; after cancelling τ, a board distinguishing two candidate input positions recovers σ. Reflection differs from identity already on the constant north board. The source permutation set therefore equals {Id,R}. The odd-size bridge covers every odd n≥3. ∎

The zero-based coordinate translation is the bijection (a,b)↦(a−1,b−1) from the source's one-based grid; it preserves all signed step equations and takes the two named roots to s,c. The implementation uses this coordinate model directly rather than formalizing a second one-based graph structure.

Certificate and limitations

Production dependency order: Board, Profiles, Reflection, Cardinals, Rigidity, Corners, Normalized, Separation, TotalSeparation, RootedCases, Endpoint, Action, AllSizes. The first eight supplied paid modules include Audit; the scientific dependency order above omits that diagnostic module. New scientific modules were compiled and separately checked with the pinned Lean kernel checker. ExpandedAudit covers all 117 declared objects and prints expanded definitions and source endpoints. Only propext, Classical.choice and Quot.sound occur transitively. Specific imports only; no added axioms, sorry, unsafe, native decision procedure, external oracle or finite-size extrapolation.

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