The Q-index of K_{t,s+1}-free graphs

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A proof of Conjecture 4 of de Freitas, Nikiforov and Patuzzi for all t >= 2 and s >= t-1, with a kernel-checked Lean 4 certificate.

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Abstract

Let q(G) be the Q-index, the largest eigenvalue of the signless Laplacian Q(G) = A(G) + D(G). Conjecture 4 of de Freitas, Nikiforov and Patuzzi asserts that a K_{t,s+1}-free graph G of order n, with s >= t-1 >= 1 and n large, satisfies q(G) <= n/2 + s + t - 2 + (1/2) sqrt((n-2+2s)^2 - 8s(n-2) + 4(t-1)(n-t+1)), with equality, among such K_{t,s+1}-free G, exactly for joins of K_{t-1} and an s-regular graph of order n-t+1. We prove this for every t >= 2 and s >= t-1, with a threshold depending only on s and t, and record a kernel-checked Lean 4 certificate for the literal statement. The proof combines a quadratic comparison, a join lemma adapted from Proposition 6 of the source, a double count bounding the number of hubs, a general hub lemma, and an integer-gap bootstrap forcing t-1 universal vertices.

1. Introduction

All graphs are finite and simple. For a graph G, V(G) is its vertex set, E(G) its edge set, n = |V(G)| its order, and d(v) the degree of a vertex v. The adjacency matrix is A(G), the degree matrix is the diagonal matrix D(G), and the signless Laplacian is Q(G) = A(G) + D(G). The Q-index q(G) is the largest eigenvalue of Q(G); since Q(G) is real symmetric, q(G) is real and equals the maximum of the Rayleigh quotient x^T Q(G) x / x^T x over nonzero real vectors x. A graph is K_{t,s+1}-free if it contains no subgraph isomorphic to the complete bipartite graph with parts of sizes t and s+1. Equivalently, every set of t vertices has at most s common neighbours: a common neighbour of such a set lies outside it, so s+1 common neighbours together with the set form a K_{t,s+1}, and conversely.

The following is Conjecture 4 in Section 1 of Maria Aguieiras A. de Freitas, Vladimir Nikiforov and Laura Patuzzi, Maxima of the Q-index: graphs with no K_{s,t}, arXiv:1507.00625v1 (2 Jul 2015). MathDB record 330858 restates it in the same scope.

Conjecture 4. Let s >= t-1 >= 1, and let n be sufficiently large. If G is a K_{t,s+1}-free graph of order n, then

q(G) <= n/2 + s + t - 2 + (1/2) sqrt((n-2+2s)^2 - 8s(n-2) + 4(t-1)(n-t+1)).

Equality holds if and only if G is a join of K_{t-1} and an s-regular graph of order n-t+1.

The source proves the case t = 2 for all s >= 1 and n >= s^2 + 6s + 6 as Theorem 5, and states that it cannot prove or disprove the conjecture for all s and t. The present paper proves the conjecture for every t >= 2 and s >= t-1, with a threshold depending only on s and t. The proof uses Proposition 6 of the source, which treats the join K_1 join H with Delta(H) <= s, with K_1 replaced by K_{t-1}; this adaptation is Lemma 1 below. It does not use Theorem 8 of the source, nor the t = 2 theorem as a black box.

Notation. Throughout, m = n-t+1, A0 = n+t-3, B0 = 2s+t-1, and f(y) = (y-A0)(y-B0) - m(t-1). We write q* for the larger root of f.

2. The theorem

Theorem 1. Let s and t be integers with 2 <= t and t-1 <= s. There is a number N, depending only on s and t, such that for every n >= N and every K_{t,s+1}-free graph G of order n,

q(G) <= q* := n/2 + s + t - 2 + (1/2) sqrt((n-2+2s)^2 - 8s(n-2) + 4(t-1)(n-t+1)),

and, among such G, equality holds if and only if G is a join of K_{t-1} and an s-regular graph of order n-t+1.

Here q* is the printed right-hand side. The equality clause is read inside the freeness hypothesis: the theorem asserts that a K_{t,s+1}-free graph G attains the bound exactly when it has the stated join form, not that every graph attaining the bound is free or has that form, and not that every join of K_{t-1} with an s-regular graph is K_{t,s+1}-free. This is the reading of the source sentence, whose Theorem 5 has the same shape.

Remark 2.1 (scope of the equality clause). The unrestricted reading, "for every graph G of order n, q(G) equals the bound if and only if G is a join of K_{t-1} and an s-regular graph", is false. At s = t = 2 and n = 4k with k >= 2, take disjoint sets A and B with |A| = |B| = 2k, put a 3-regular graph on A, a perfect matching on B, and a (2k-2)-regular bipartite graph between A and B. The partition is equitable for Q(G) with quotient matrix [[6+m, m], [m, 2+m]], where m = 2k-2; its trace is n+4 and its determinant is 4(n-1), so its larger eigenvalue is the printed bound at s = t = 2, and the quotient's Perron vector lifts to a positive eigenvector. No vertex has degree n-1, so G is not a join with K_1. This witness is proved in Lean at n = 8, where q(G) = 6 + 2 sqrt 2 equals the printed bound at (n,s,t) = (8,2,2); the general family is a hand argument checked numerically for n = 8, 12, ..., 40. The reading "every join of K_{t-1} with an s-regular graph is K_{t,s+1}-free and extremal" is also false: K_1 join C_4 is the wheel, a join of K_1 with a 2-regular graph, and it contains K_{2,3}. A third statement, that a join of K_{t-1} with an s-regular graph attains the bound, is true without the freeness hypothesis, but it is not part of Theorem 1.

3. The quadratic and the join lemma

Set A0 = n+t-3, B0 = 2s+t-1, m = n-t+1, and f(y) = (y-A0)(y-B0) - m(t-1). Let q* be the larger root of f.

Lemma 0. (i) f(max(A0,B0)) = -m(t-1) < 0, so q* > max(A0,B0). (ii) f(n+2t-4) = -2(t-1)(s-t+2) < 0 whenever s >= t-1, so q* > n+2t-4. (iii) If s = t-2 then q* = n+2t-4. The hypothesis s >= t-1 is used only in (ii), to make the inequality strict; the boundary case s = t-2 is exactly where it fails.

Proof. (i) At y=A0 and at y=B0 the value is -m(t-1), which is negative because m = n-t+1 >= 1. Hence max(A0,B0) lies between the two roots, so q* > max(A0,B0). (ii) f(n+2t-4) = (n+2t-4-(n+t-3))(n+2t-4-(2s+t-1)) - m(t-1) = (t-1)(n+t-3-2s) - (t-1)(n-t+1) = (t-1)(2t-4-2s) = -2(t-1)(s-t+2). Since s >= t-1, s-t+2 >= 1, so this is negative, and n+2t-4 lies between the roots; hence q* > n+2t-4. (iii) If s = t-2, the same computation gives f(n+2t-4)=0, so n+2t-4 is a root. Since n+2t-4 > max(A0,B0) for n > t-1, it is the larger root, so q* = n+2t-4. Finally, the printed right-hand side is q*: (A0+B0)/2 = n/2+s+t-2, and (A0-B0)^2 + 4m(t-1) = (n-2-2s)^2 + 4(t-1)(n-t+1) = (n-2+2s)^2 - 8s(n-2) + 4(t-1)(n-t+1), so the larger root is exactly the printed expression.

Lemma 1 (join lemma). Let t >= 2 and s >= 0, let H be a graph of order m = n-t+1 >= 1 with Delta(H) <= s, and let G = K_{t-1} join H. Then q(G) <= q*, with equality if and only if H is s-regular.

An earlier version of this lemma carried a loose hypothesis, written with an ellipsis, requiring n to be large enough that q* > B0. That hypothesis is unnecessary and has been removed: f(max(A0,B0)) = -m(t-1) < 0 for every n > t-1, so q* > max(A0,B0) always.

Proof. By the Perron-Frobenius theorem for nonnegative matrices, Q(G) has a nonzero eigenvector x >= 0 for its

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largest eigenvalue q = q(G). If q <= max(A0,B0), then q < q* by Lemma 0, and the bound is strict. Assume q > max(A0,B0). Let a = max over w in U of x_w, where U = V(K_{t-1}), and b = max over u in V(H) of x_u, attained at w and u. At w, d(w)=n-1 and N(w) = (U minus {w}) union V(H), so

q a = (n-1)a + sum over w' in U minus {w} of x_w' + sum over v in V(H) of x_v <= (n-1)a + (t-2)a + m b = A0 a + m b,

hence (q-A0)a <= m b. At u, d(u) = d_H(u)+t-1, and

q b = (d_H(u)+t-1)b + sum over v in N_H(u) of x_v + sum over w in U of x_w <= (d_H(u)+t-1)b + d_H(u)b + (t-1)a = (2d_H(u)+t-1)b + (t-1)a <= (2s+t-1)b + (t-1)a = B0 b + (t-1)a,

hence (q-B0)b <= (t-1)a. If b=0, the first inequality gives a=0 because q-A0>0; if a=0, the second gives b=0 because q-B0>0. Since x is nonzero, a,b>0. Multiplying the two inequalities gives (q-A0)(q-B0) <= m(t-1), i.e., f(q) <= 0. Since q > max(A0,B0), this implies q <= q*. So q(G) <= q* in all cases.

If q = q*, then f(q)=0 and both inequalities are equalities, all factors being positive. Equality in the first forces x=b on V(H) and x=a on U. For any v in V(H), the eigenequation now reads q b = (2d_H(v)+t-1)b + (t-1)a, while (q-B0)b = (t-1)a gives q b = (2s+t-1)b + (t-1)a; hence d_H(v)=s for every v in V(H), i.e., H is s-regular.

Conversely, if H is s-regular, choose a,b>0 with (q*-A0)a = m b and (q*-B0)b = (t-1)a; this is possible because q* > max(A0,B0) and f(q*)=0. The vector equal to a on U and b on V(H) is a positive eigenvector of Q(G) with eigenvalue q*. Hence q* is an eigenvalue, so q(G) >= q*. Combined with q(G) <= q*, this gives q(G)=q*.

4. Few hubs and the hub lemma

Lemma 2 (few hubs). Let t >= 2, s >= 0, D >= 1, and l0 = tD. Let G be a graph of order n in which every set of t vertices has at most s common neighbours. If n > s*l0*binom(l0,t), then fewer than l0 vertices of G have degree at least n/D.

Proof. Suppose, for contradiction, that L0 is a set of l0 vertices each of degree at least n/D. For v in V(G), let d_{L0}(v) = |N(v) intersect L0|. Then sum over v in V(G) of d_{L0}(v) = sum over w in L0 of d(w) >= l0 n/D = t n. Vertices with d_{L0}(v) <= t-1 contribute at most t-1 each, so their total contribution is at most (t-1)n. Hence the sum over vertices with d_{L0}(v) >= t is at least n. Each such vertex has d_{L0}(v) <= l0, so there are at least n/l0 of them. For each such v choose a t-subset T of L0 contained in N(v) intersect L0; this is possible because d_{L0}(v) >= t. There are binom(l0,t) possible t-subsets, so by the pigeonhole principle some t-subset T is chosen by at least (n/l0)/binom(l0,t) vertices. Since n > s*l0*binom(l0,t), this number is greater than s, so T has at least s+1 common neighbours, contradicting the hypothesis. The only property of K_{t,s+1}-freeness used is that every set of t vertices has at most s common neighbours.

Lemma 3 (hub lemma). Let G be any graph of order n with q = q(G) >= n and a Q-eigenvector x >= 0 with max over v of x_v = 1. Fix delta in (0,1/4] and epsilon in (0,1). Suppose at most l vertices of G have degree at least delta n, and let L be the set of those vertices. Put P = {v : d(v) >= (1-epsilon)n}, p = |P|, T = V(G) minus P, S = sum over v in P of x_v, and sigma = sum over v in T of x_v. Let C = max(2l, 3l/epsilon). Assume n > C. Then:

(a) x_v <= C/n for every v outside P; in particular the vertex with x_v=1 lies in P, so p >= 1 and S >= 1.

(b) sigma <= S + 4 delta S + 4Cl/n.

(c) for every u in P, (q-d(u)+1)x_u <= 2S + eta, where eta = 4 delta S + 4Cl/n.

Consequently q <= n + 2p - 2 + eta.

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Proof. (a) Let y = max over v not in L of x_v, attained at v0. Under n > C >= 2l, the set V(G) minus L is nonempty. The eigenequation at v0 gives (q-d(v0))y = sum over w in N(v0) of x_w <= l + d(v0)y, so (q-2d(v0))y <= l. Since d(v0) < delta n, q-2d(v0) >= n-2delta n = (1-2delta)n, and y <= l/((1-2delta)n) <= 2l/n because delta <= 1/4. For u in L minus P, (q-d(u))x_u = sum over w in N(u) of x_w <= l + n y <= 3l, while q-d(u) >= n-(1-epsilon)n = epsilon n; hence x_u <= 3l/(epsilon n) <= C/n. For v outside P and outside L, x_v <= y <= 2l/n <= C/n. This proves (a). Since C/n < 1, the vertex with x_v=1 is not outside P, so it lies in P; hence p>=1 and S>=1.

(b) Summing the eigenequation over T gives

sum over v in T of (q-d(v))x_v = sum over w in P of d_T(w)x_w + sum over v in T of d_T(v)x_v,

so

sum over v in T of (q-d(v)-d_T(v))x_v = sum over w in P of d_T(w)x_w <= nS.

For v in T minus L the coefficient q-d(v)-d_T(v) is at least q-2delta n. For v in T intersect L it is at least q-2n, and x_v <= C/n with at most l such vertices. Hence

(q-2delta n) sigma - 2Cl <= nS.

Since q >= n, q-2delta n >= (1-2delta)n, so

sigma <= (S + 2Cl/n)/(1-2delta) <= (1+4delta)(S+2Cl/n) <= S + 4delta S + 4Cl/n,

where we used delta <= 1/4.

(c) For u in P, (q-d(u))x_u = sum over w in N(u) of x_w <= (S - x_u) + sigma <= 2S - x_u + eta, so (q-d(u)+1)x_u <= 2S + eta. For the consequence, take u with x_u=1, which lies in P by (a). Since d(u) <= n-1, q <= d(u)-1+2S+eta <= n-1-1+2p+eta = n+2p-2+eta. Lemma 3 uses no property of K_{t,s+1}-freeness; it is a statement about any graph with few vertices of degree at least delta n.

5. Proof of the theorem

Let D = 16t^2, delta = 1/D, l0 = tD = 16t^3, l = l0, epsilon = 1/(2t), and C = max(2l0, 3l0/epsilon) = 6t l0. Choose N so large that

n > s*l0*binom(l0,t), n > 2s, n > C, and 4C l0/n <= 1/(4t).

These conditions depend only on s and t. Let G be a K_{t,s+1}-free graph of order n >= N.

Case 1: q(G) < n+2t-4. By Lemma 0, q(G) < q*. Also G is not a join of K_{t-1} and an s-regular graph of order n-t+1: such a join has q = q* by Lemma 1. Hence the bound is strict and equality does not hold.

Case 2: q := q(G) >= n+2t-4. Then q >= n. By the Perron-Frobenius theorem Q(G) has a nonnegative eigenvector x for q, normalized so that its largest entry is 1. Lemma 2, with D and l0 as above, shows that fewer than l0 vertices have degree at least delta n. Apply Lemma 3 with delta, epsilon, l = l0 and C. Its hypotheses hold: delta = 1/(16t^2) <= 1/4, epsilon = 1/(2t) in (0,1), and n > C. Let P, p, S, T, sigma be as in Lemma 3, and eta = 4delta S + 4Cl/n.

First, p <= t-1. Indeed, if p >= t, then t vertices of P have degree at least (1-epsilon)n = n - n/(2t), so they have at least n - t*n/(2t) = n/2 common neighbours. Since n > 2s, n/2 > s, so these t vertices have at least s+1 common neighbours, contradicting K_{t,s+1}-freeness.

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Since S <= p <= t-1, eta <= 4delta(t-1) + 4Cl/n <= (t-1)/(4t^2) + 1/(4t) <= 1/(2t). Lemma 3 gives n+2t-4 <= q <= n+2p-2+eta, so 2p >= 2t-2-eta and p >= t-1-eta/2 > t-2. Since p is an integer, p = t-1.

Let u in P with x_u=1, which exists by Lemma 3(a). From Lemma 3(c), q-d(u)+1 <= 2S+eta. Since q >= n+2t-4 and d(u) <= n-1, the left side is at least 2t-2, so S >= t-1-eta/2. Hence sum over w in P minus {u} of x_w = S-1 >= t-2-eta/2. There are t-2 terms in this sum, each at most 1, so each x_w >= 1-eta/2 for w in P.

Now fix w in P. Lemma 3(c) gives (q-d(w)+1)(1-eta/2) <= 2S+eta <= 2(t-1)+eta, so

q-d(w)+1 <= (2t-2+eta)/(1-eta/2) < 2t-1,

because eta(2t+1)/2 < 1, which follows from eta <= 1/(2t). Hence d(w) > q+2-2t >= n-2, so d(w) >= n-1. Since d(w) <= n-1, d(w)=n-1. Thus every vertex of P is universal, and P induces K_{t-1}.

Write H = G minus P. Then H has order n-(t-1) = n-t+1, and G is the join of K_{t-1} and H. If some v in H had s+1 neighbours in H, then P union {v} would be a set of t vertices with s+1 common neighbours, a contradiction. Hence Delta(H) <= s. Lemma 1 gives q <= q*, with equality if and only if H is s-regular.

This proves the bound in both cases. For the equality clause: if q=q*, then q > n+2t-4, so Case 2 applies and G is a join of K_{t-1} and an s-regular graph H of order n-t+1. Conversely, if G is K_{t,s+1}-free and is such a join, then H is s-regular, so Lemma 1 gives q(G)=q*. This proves Theorem 1.

6. Where the Lean proof differs

The written proof uses the Perron-Frobenius theorem to obtain a nonnegative eigenvector for q. The Lean development avoids Perron-Frobenius. It takes a top eigenvector y, forms z = |y|, and uses the positive semidefiniteness of qI - Q to show that z is again a top eigenvector. The relevant declarations are exists_top_eigenvector, top_sub_posSemidef and exists_nonneg_top_eigenvector in QIdxSpectral.

The Lean endpoint uses an explicit threshold, thresholdN s t, which is quoted below. The written proof only asserts that a threshold exists and uses the sufficient conditions listed in Section 5; it does not claim that thresholdN is optimal or that the Lean threshold coincides with the threshold N used here.

Independence caveat. Two hand-written versions of this proof exist, and the hub lemma (Lemma 3) shares its mechanism with the structural proposition of the other version. The version here differs only in using a split at delta n in place of an edge count. It is not an independent method.

```
def thresholdN (s t : â) : â :=
  (t * (16 * t ^ 2)) * (s * (t * (16 * t ^ 2)).choose t) + 2 * s
    + 6 * t * (3 * (t * (16 * t ^ 2)) * (2 * t)) * (t * (16 * t ^ 2))
    + 3 * (t * (16 * t ^ 2)) * (2 * t) + t + 1
```

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7. Lean certificate

The certificate is for Lean 4.34.1 with mathlib commit d13f23b723b8a846827a245b89c10fc7d3f11612. The development consists of nine modules: QIdxStatement, QIdxCounting, QIdxQuadratic, QIdxSpectral, QIdxEstimates, QIdxJoin, QIdxBootstrap, QIdxBridge and QIdxEndpoint. The endpoint is the theorem qindex_kts_endpoint, whose type is the literal statement Statement.

In the quotations below, the characters for the real numbers, the natural numbers and the top element are printed as R, N and T, and the graph-isomorphism symbol is printed as a congruent sign, because the installed groff PostScript fonts have no glyphs for the original characters; subscripts in identifiers are printed as ordinary digits, so the two graph identifiers in graphJoin appear as G1 and G2; the angle brackets of anonymous constructors are printed as parentheses, and the middle dot is printed as a period. The source file carries the exact Unicode tokens. The definitions used in the statement are the following, quoted exactly from QIdxStatement.lean:

```
noncomputable def signlessLaplacian (G : SimpleGraph V) [DecidableRel G.Adj] : Matrix V V â :=
  G.adjMatrix â + G.degMatrix â

noncomputable def qIndex (G : SimpleGraph V) [DecidableRel G.Adj] : â :=
  sSup (spectrum â (signlessLaplacian G))

def graphJoin {A B : Type*} (Gâ : SimpleGraph A) (Gâ : SimpleGraph B) : SimpleGraph (A â B) where
  Adj x y :=
    match x, y with
    | Sum.inl a, Sum.inl a' => Gâ.Adj a a'
    | Sum.inr b, Sum.inr b' => Gâ.Adj b b'
    | Sum.inl _, Sum.inr _ => True
    | Sum.inr _, Sum.inl _ => True
  symm := â¨by
    rintro (a | b) (a' | b') h
    Â. exact (show Gâ.Adj a a' from h).symm
    Â. trivial
    Â. trivial
    Â. exact (show Gâ.Adj b b' from h).symmâ©
  loopless := â¨by
    rintro (a | b) h
    Â. exact Gâ.loopless.irrefl a h
    Â. exact Gâ.loopless.irrefl b hâ©

noncomputable def printedBound (n s t : â) : â :=
  (n : â) / 2 + s + t - 2 +
    (1 / 2) * Real.sqrt (((n : â) - 2 + 2 * s) ^ 2 - 8 * s * ((n : â) - 2)
      + 4 * ((t : â) - 1) * ((n : â) - t + 1))

def Statement : Prop :=
  â s t : â, 2 â¤ t â t - 1 â¤ s â
    â N : â, â n : â, N â¤ n â â (G : SimpleGraph (Fin n)) [DecidableRel G.Adj],
      (completeBipartiteGraph (Fin t) (Fin (s + 1))).Free G â
        qIndex G â¤ printedBound n s t â§
        (qIndex G = printedBound n s t
          â (H : SimpleGraph (Fin (n - t + 1))) (_ : DecidableRel H.Adj),
            H.IsRegularOfDegree s â§
              Nonempty (G âg graphJoin (â¤ : SimpleGraph (Fin (t - 1))) H))

theorem qIndex_isGreatest [Nonempty V] (G : SimpleGraph V) [DecidableRel G.Adj] :
    IsGreatest (spectrum â (signlessLaplacian G)) (qIndex G) := by
  have hrange := (signlessLaplacian_isHermitian G).spectrum_real_eq_range_eigenvalues
  have hfin : (spectrum â (signlessLaplacian G)).Finite := by
    rw [hrange]; exact Set.finite_range _
  have hne : (spectrum â (signlessLaplacian G)).Nonempty := by
    rw [hrange]; exact Set.range_nonempty _
  exact â¨hne.csSup_mem hfin, fun _ hx => le_csSup hfin.bddAbove hxâ©
```

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The endpoint theorem, quoted exactly from QIdxEndpoint.lean, is:

```
theorem qindex_kts_endpoint : Statement := by
  intro s t ht hs
  refine â¨thresholdN s t, ?_â©
  intro n hn G _ hfree
  have hcard : Fintype.card (Fin n) = n := Fintype.card_fin n
  have hmain := main_general G ht hs (by rw [hcard]; exact hn) hfree
  rw [hcard] at hmain
  exact hmain
```

The audit output records:

'QIdx.main_general' depends on axioms: [propext, Classical.choice, Quot.sound]

'QIdx.qindex_kts_endpoint' depends on axioms: [propext, Classical.choice, Quot.sound]

The modelling decisions that matter are these. The scalars are natural numbers s and t, with 2 <= t and t-1 <= s; the threshold is existentially quantified after s and t and before n. Graphs are SimpleGraph (Fin n) with a DecidableRel instance, which covers every graph of order n up to relabelling. Q is adjMatrix R + degMatrix R. qIndex is the supremum of the real spectrum of Q, and qIndex_isGreatest proves it is the greatest element. Freeness is mathlib's Free, i.e., absence of an injective graph homomorphism from the complete bipartite graph, so it is ordinary non-induced containment. printedBound uses real arithmetic throughout, including real subtraction in n-t+1. graphJoin is defined by hand as the disjoint union with all cross edges; K_{t-1} is the complete graph on Fin (t-1); H is a graph on Fin (n-t+1) with IsRegularOfDegree s. The equality clause is placed inside the freeness hypothesis, as in the source sentence and its Theorem 5.

Independent audit. The independent audit rebuilt all nine modules from an empty build and kernel-checked each one; it read every module; it walked the 35,041 constants in the transitive closure of the two main theorems and found no missing constant; the axioms are exactly propext, Classical.choice and Quot.sound. It proved semantic controls: the triangle has q = 4 and the printed bound at (3,1,2) is 4; the join form of the triangle; K_4 is not C_4-free, and an induced reading would make this false; a free graph (the bowtie) is accepted; four variants of the formula (a sign change, a constant change, the order change n-t for n-t+1, and a truncated subtraction) differ from printedBound; the path on three vertices has q = 3; an explicit no-injective-map form of freeness is equivalent to Free; free_swap; the bowtie's q and bound are (7+sqrt 17)/2; equality for the wheel K_1 join C_4; the scope witnesses of Remark 2.1; and two negative controls, a false claim and a sorry, were rejected with exit 1. The audit trusted upstream Lean and mathlib: it verified hashes of Mathlib.lean and lake-manifest.json against PINS and Lean's version string, not the mathlib revision itself, and not every olean. It compared Conjecture 4 against the text layer of the source, not the page image. The audit was performed by an AI system of the same family as the proof authors, not by a human referee. Its independence was limited: it read the proof and the earlier AI reviews before reaching its view, so this is verification, not independent invention. The audit recorded four defects: a cosmetic docstring reference to a file not in the source-only packet; a low-severity absence of semantic controls beyond qIndex_isGreatest, which the audit supplied; and two informational items, a heartbeat option and the placement of the unconditional reverse direction under freeness. None affects validity or fidelity.

Status. The audit recommended a technical pass and reported no defect affecting validity or fidelity. The receipts, hashes and mathlib pin of the certificate were verified separately. This is not a human referee report.

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8. Limits

The prior-art search behind this paper was bounded, and no citation index was available. Four works were not read: (i) a UFRGS thesis, Indices de grafos livres de K_{s,t}; its title concerns indices of K_{s,t}-free graphs, and if it proves the conjecture then this result is not new; (ii) the final journal version of the source (LAA 496), which was not read; (iii) the body of Yu, Wang and Guo, ELA 30 (2015) 795-811; and (iv) RAIRO Operations Research 10.1051/ro/2023109, which could not be retrieved (HTTP 403) and whose title was never seen.

The nearest printed results read for this paper are the upper bounds of Kong and Wang [2] and the minor-free results of Chen and Zhang [3]; neither gives the exact bound of Theorem 1. These are context, not a novelty clearance. The general witness family in Remark 2.1 is a hand argument checked numerically to n = 40; only the n = 8 case is in Lean. It is a side remark and is not used in the proof of Theorem 1.

Objections to significance. The AI reviews of this work state: the join lemma is a routine adaptation of Proposition 6 of the source; the structural step follows a heavy-vertex template known in adjacency work; the proof is short; and the significance was judged modest. No peer review, priority or worldwide novelty is claimed.

References

[1] Maria Aguieiras A. de Freitas, Vladimir Nikiforov and Laura Patuzzi, Maxima of the Q-index: graphs with no K_{s,t}, arXiv:1507.00625v1, 2 Jul 2015.

[2] Qi Kong and Ligong Wang, Upper bounds on the Q-spectral radius of book-free and/or K_{s,t}-free graphs, arXiv:1612.03514v1, 12 Dec 2016.

[3] Ming-Zhu Chen and Xiao-Dong Zhang, On the signless Laplacian spectral radius of K_{s,t}-minor free graphs, arXiv:1908.04221v1, 12 Aug 2019.

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