The list color function threshold of K_{2,l}

A proof that Ï(K_{2,l}) = Î(âl), with a Lean 4 certificate

Myeongjin Kang. Prepared with AI assistance.

Abstract

We prove that for every m â¥ 4 and every l â¥ 1 with 2l â¤ (mâ1)^2, each m-assignment of the complete bipartite graph K_{2,l} admits at least P(K_{2,l},m) = m(mâ1)^l + m(mâ1)(mâ2)^l proper list colourings. This gives Ï(K_{2,l}) â¤ max(4, ceil(1 + â(2l))), which together with the lower bound of Kaul, Kumar, Mudrock, Rewers, Shin and To determines the order of growth conjectured in that paper: Ï(K_{2,l}) = Î(âl). The argument is about a page: Bernoulli on the diagonal of the shared list colours, one AM-GM over a single block, and a scalar inequality closed by two estimates. The literal statement is also proved in Lean 4 against mathlib, and the kernel check uses only propext, Classical.choice and Quot.sound. This is a draft; it has not been refereed or published, and no priority is claimed.

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1. Introduction

Throughout, G is a finite simple graph and m is a positive integer. The chromatic polynomial P(G,m) counts the proper colourings of G with colours from a fixed set of size m. A list assignment L gives each vertex v a list L(v); it is an m-assignment if |L(v)| = m for every v. P(G,L) counts the proper colourings f with f(v) â L(v), and the list colour function is P_â(G,m) = min P(G,L), the minimum over all m-assignments L. Since the same m colours may be given to every vertex, P_â(G,m) â¤ P(G,m) always.

Section 1.2 of [1] introduces two thresholds. The list color function number Î½(G) is the smallest t â¥ Ï(G) such that P_â(G,t) = P(G,t). The list color function threshold Ï(G) is the smallest k â¥ Ï(G) such that P_â(G,m) = P(G,m) whenever m â¥ k. The second definition quantifies over every larger number of colours, not only equality at m = k, and that is the difference between Ï and Î½. Both are well defined by Donner's 1992 result, as [1] records. For the graph at issue, [1] records

P(K_{2,l}, m) = m(mâ1)^l + m(mâ1)(mâ2)^l,

and states Conjecture 5: Ï(K_{2,l}) = Î(âl) as l â â. The lower half is Theorem 4 of [1], quoted exactly in Section 4 below. The question is recorded as MathDB 353552.

The upper half is what was missing. The follow-up [2] proves Ï(K_{2,n}) â¤ ceil((n+2.05)/1.24), a linear bound, and in its Theorem 7 it verifies by machine that P_â(K_{2,n},m) = P(K_{2,n},m) for m = 3 and 2 â¤ n â¤ 10, for m = 4 and 2 â¤ n â¤ 24, and for m = 5 and 2 â¤ n â¤ 43, with strict inequality from n = 12, 27 and 44, and n = 11, 25, 26 open. Its page 4 says of Conjecture 4, its restatement of Conjecture 5: "it seems that a proof of Conjecture 4 will require a novel refinement of our approach or an entirely new approach."

What [2] already contains matters for what is new here. Lemma 5 of [2] is the containment reduction: some minimising assignment has every y-list inside the union of the two x-lists. Lemma 6 of [2] is the case of exactly one shared colour, with precisely the Bernoulli trade-off between the diagonal gain and the loss on the two remaining colours. Lemma 13 of [2] is a block AM-GM for the case of mâ2 shared colours, and its fourth term already isolates the second-order loss. The proof of Lemma 15 of [2] contains the joint-miss AM-GM, two-variable, on the counts of colours missed in two blocks. The present argument keeps the diagonal and one block apart for every number of shared colours, bounds the joint-miss loss uniformly, and closes with a scalar inequality valid up to l of order m^2; that uniform step is what [2] does not have.

2. The results

Theorem 1. Let m â¥ 4 and l â¥ 1 be integers with 2l â¤ (mâ1)^2. Then every m-assignment L of K_{2,l} satisfies

P(K_{2,l}, L) â¥ P(K_{2,l}, m) = m(mâ1)^l + m(mâ1)(mâ2)^l.

Consequently P_â(K_{2,l}, m) = P(K_{2,l}, m).

Theorem 2. Ï(K_{2,l}) â¤ max(4, ceil(1 + â(2l))).

Corollary 3. Ï(K_{2,l}) = Î(âl) as l â â.

3. Proof of Theorem 1

3.1 Notation. Let the parts of G = K_{2,l} be {x1,x2} and {y1,...,yl}. Fix an m-assignment L and put A = L(x1), B = L(x2) and S_j = L(y_j); all have size m. Let D = A â© B, d = |D|, k = mâd, A' = A \ D and B' = B \ D, so that |A'| = |B'| = k and A' and B' are disjoint. Put

t = mâ1, s = (mâ2)/t, q = 1 + 1/t, Îº = s q = 1 â 1/t^2.

Then 0 < s < 1 < q and Îº^(1/4) â¥ s, since Îº = sq â¥ s^2 â¥ s^4.

For a colour c let u_j(c) be 1 if c â S_j and 0 otherwise, and let M(c) = â_j u_j(c). The vertices x1 and x2 are not adjacent, so a = b is allowed, and

P(G,L) = â_{a â A} â_{b â B} â_j |S_j \ {a,b}|.

For a = b the j-th factor is mâ1+u_j(a) = t(1 + u_j(a)/t); for a â b it is mâ2+u_j(a)+u_j(b) = t(s + (u_j(a)+u_j(b))/t). Dividing by t^l and writing

W(a,a) = â_j (1 + u_j(a)/t), W(a,b) = â_j (s + (u_j(a)+u_j(b))/t)

we get

P(G,L)/t^l = â_{a â D} W(a,a) + â_{a â b} W(a,b), P(G,m)/t^l = m + m t s^l.

Also W(a,a) â¥ 1 + M(a)/t, and W(a,b) â¥ s^l for a â b.

3.2 Step 1: reduction to the block A' Ã B'. Every off-diagonal pair has W(a,b) â¥ s^l, and A Ã B has m^2âd off-diagonal pairs, of which k^2 lie in A' Ã B'. With Î = â_{c â D} M(c), and using the diagonal bound,

P(G,L)/t^l â¥ d + Î/t + â_{a â A', b â B'} W(a,b) + (m^2âdâk^2) s^l.

Since m = d+k, we have dâm = âk and m^2âdâk^2âmt = kâk^2. Subtracting the target m + m t s^l,

P(G,L)/t^l â (m + m t s^l) â¥ Î/t + â_{A' Ã B'} W â k â k(kâ1) s^l.

For k = 0 the right side is nonnegative. For k â¥ 1 it is enough to prove

(I) Î/t + â_{a â A', b â B'} W(a,b) â¥ k + k(kâ1) s^l.

3.3 Step 2: counting misses. For each j put Îµ_j = |A' \ S_j|, Î²_j = |B' \ S_j| and Î´_j = |D \ S_j|. Since |A âª B| = m+k while |S_j| = m, and the three sets of missed colours are disjoint,

Îµ_j + Î²_j + Î´_j â¥ k for every j, and Î = â_j Î´_j.

3.4 Step 3: the case k = 1. Here A' = {a} and B' = {b} with a â b. For each j, either u_j(a)+u_j(b) â¥ 1, and then the j-th factor of W(a,b) is at least s + 1/t = 1, or u_j(a) = u_j(b) = 0, and then the factor is s while Î´_j â¥ 1. If e is the number of indices of the second kind, then W(a,b) â¥ s^e and Î â¥ e, so by Bernoulli

Î/t + W(a,b) â¥ e/t + (1 â 1/t)^e â¥ 1,

which is (I) for k = 1.

3.5 Step 4: k â¥ 2, an averaged joint-miss bound. By AM-GM over the k^2 pairs of A' Ã B',

â_{A' Ã B'} W(a,b) â¥ k^2 â_j G_j,

where G_j is the geometric mean of the k^2 numbers s + (u_j(a)+u_j(b))/t. With x = Îµ_j/k and y = Î²_j/k, a fraction xy of the pairs misses both colours, a fraction x+yâ2xy misses exactly one, and a fraction (1âx)(1ây) misses neither, so

G_j = q^(xy) s^((1âx)(1ây)),

which is nondecreasing in x and in y. Let Î¶_j = min(1, Î´_j/k), lying in [0,1]. By Step 2, x+y â¥ 1âÎ¶_j. Lowering x and y until their sum is w = 1âÎ¶_j, and using (1âx)(1ây) = Î¶_j + xy with xy â¤ w^2/4 â¤ w/4, we obtain

G_j â¥ Îº^(w/4) s^(Î¶_j) = Îº^((1âÎ¶_j)/4) s^(Î¶_j),

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because Îº < 1. With Z = â_j Î¶_j â [0,l] and B(Z) = Îºˆ((lâZ)/4) sˆZ,

â_{A' Ã B'} W â¥ kˆ2 B(Z) and Î â¥ k Z.

So (I) follows from Z/t + k B(Z) â¥ 1 + (kâ1) sˆl. Now B(Z) â¥ sˆ(lâZ) sˆZ = sˆl because Îºˆ(1/4) â¥ s, so k B(Z) â (kâ1) sˆl is nondecreasing in k; hence k = 2 is the worst case, and it is enough to prove

(III) Z/t + 2 B(Z) â¥ 1 + sˆl for every real Z â [0,l].

3.6 Step 5: the scalar inequality. Put y = sˆl, v = Îºˆ(l/4), Î» = ln(Îºˆ(1/4)/s) > 0 and g = t Î», so that B(Z) = v eˆ(âÎ» Z). Since Îº = (1â1/t)(1+1/t),

g = t[(1/4) ln(1+1/t) â (3/4) ln(1â1/t)] = â_{n â¥ 1} (1/n)(3/4 + (â1)ˆ(n+1)/4) tˆ(1ân),

so g has positive coefficients, g > 1, g decreases in t, and for t â¥ 3,

g â¤ g(3) = (3/4) ln(4/3) + (9/4) ln(3/2) = 1.128058...

Case 1: y â¥ 1/(2g). With p = lâZ â [0,l] we have B(Z) = y eˆ(Î» p) â¥ y(1 + Î» p). Hence

Z/t + 2B(Z) â 1 â y â¥ (l/t + y â 1) + p(2 y Î» â 1/t) â¥ 0,

the first bracket by Bernoulli, sˆl â¥ 1 â l/t, and the second because 2 y Î» t = 2 y g â¥ 1.

Case 2: y < 1/(2g). The convex function Z â Z/t + 2 v eˆ(âÎ» Z) has minimum (ln(2vg)+1)/g over all real Z. So the left side of (III) minus 1+y is at least

(ln(2vg)+1)/g â 1 â 1/(2g) = Ï/g, Ï = ln(2vg) + 1/2 â g.

Since l â¤ tˆ2/2 and ln Îº < 0, ln v = (l/4) ln Îº â¥ (tˆ2/8) ln(1â1/tˆ2), and tˆ2 ln(1â1/tˆ2) increases in t, so ln v â¥ (9/8) ln(8/9) = â0.132507... for t â¥ 3. The map g â ln g â g decreases for g â¥ 1, so with 1 < g â¤ 1.128058,

Ï â¥ ln 2 â 0.132507 + ln(1.128058) + 1/2 â 1.128058 = 0.053079... > 0.

This proves (III), hence (I), hence Theorem 1.

Where the hypotheses are used. The bound m â¥ 4 is used only in Step 5, through t = mâ1 â¥ 3: for the monotonicity of g in t and for tˆ2 ln(1â1/tˆ2) â¥ 9 ln(8/9). The bound 2l â¤ (mâ1)ˆ2 is used only in Case 2 of Step 5, to bound ln v from below. Steps 1 to 4 and Case 1 need only m â¥ 3; the inequality Îºˆ(1/4) â¥ s needs only t â¥ 2.

Remark (a shorter variant). A shorter proof of the same theorem was found by an independent AI reviewer. It keeps only the diagonal of D and the block A' Ã B', bounds every other off-diagonal pair by (mâ2)ˆl, and uses the constraint k_i â¥ p_i + q_i â k, where k_i is the number of colours of D missed by S_i and p_i, q_i are the numbers of colours of A', B' that S_i contains. Bernoulli on the diagonal and AM-GM on the block then reduce the claim to a one-variable convex inequality, which is monotone in k and so reduces to k = 2, and is closed in two regimes. It needs neither a containment reduction nor the mixed pairs, and it proves the same theorem, though not the sharp constant of Section 5. We record it only as an independent check, not as a separate result.

4. The threshold and the lower bound

Let k0 = max(4, ceil(1 + â(2l))). For every m â¥ k0 we have m â¥ 4 and (mâ1)ˆ2 â¥ 2l, so Theorem 1 gives P_â(K_{2,l},m) â¥ P(K_{2,l},m), while the reverse inequality is trivial; hence P_â(K_{2,l},m) = P(K_{2,l},m) for every m â¥ k0. Since Ï(K_{2,l}) â¤ 2 < k0, the number k0 belongs to the defining set of Ï, so Ï(K_{2,l}) â¤ k0, which is Theorem 2 and gives Ï(K_{2,l}) = O(âl).

For the lower half we quote Theorem 4 of [1] as printed: "Suppose G = K_{2,l} and l â¥ 16. Let q = floor(l/4). Then Ï(G) > floor((q / ln(16/7))ˆ(1/2) + 1)." Consequently there is a constant C > 0 such that Ï(K_{2,l}) â Ï_â(K_{2,l}) = Ï(K_{2,l}) â 3 â¥ C âl for each l â¥ 16, so Ï(K_{2,l}) = Î©(âl). With Theorem 2 this gives Corollary 3.

In Lean the upper bound is the theorem tau_le_kB, whose proof passes through sq_of_ceil_le, kB_mem_tau_set and tau_set_nonempty; the last shows that the set defining Ï is nonempty, so the infimum is a genuine minimum and not the junk value 0. The lower bound is reproved, not quoted: tau_gt_source states, for l â¥ 16 and q = l/4 in natural-number division, floor(â(q/ln(16/7)) + 1) < Ï(K l), which is the displayed inequality of Theorem 4 for every l â¥ 16 and not only for multiples of 4. A cruder bound, tau_lower, gives âl â¤ 4 Ï(K l) for l â¥ 144 and is what the asymptotic statement uses.

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5. Two remarks

Remark 1 (sharp constant; not formalized, not part of the certified result). The same reduction, followed through the weights route with a = 1/(mâ1), reduces the remaining inequality to H â¥ 0 where H = Q(xâ1) + 2 u Î´ â 1 + 4 x âu, with Q = qˆl, u = qˆn, Î´ = rˆ(n/4), r = 1âaˆ2 and x = a t. For each Î¸ < 4 ln 2 one shows H â¥ 0 whenever aˆ2 l â¤ Î¸, in two regimes: if u â¥ 1/B with B = eˆ(âÎ¸/4) â 1/2 > 0 then H â¥ 0 directly, and otherwise n a is bounded by ln(1/B), which makes the two factors explicit. This gives Ï(K_{2,l}) â¤ 2 + â(l/Î¸), hence limsup Ï/âl â¤ 1/âÎ¸, and letting Î¸ tend to 4 ln 2 gives limsup â¤ 1/(2 â(ln 2)). For the matching lower bound, Lemma 12 of [1], with Îµ tending to 0, gives Ï(K_{2,l}) > m once l â¥ 4t with t = floor(l/4) and mâ1 of order â(l/Î¸) for any Î¸ > 4 ln 2, hence liminf â¥ 1/(2 â(ln 2)). Therefore Ï(K_{2,l})/âl â 1/(2 â(ln 2)). This is a remark about the same family, checked by hand by two AI reviewers: it is not formalized in Lean and is not part of the certified statement, which is the Î class only.

Remark 2 (a hand-checked note for K_{3,l}). In fact Ï(K_{3,l}) = Î©(l/log l), as follows. Take m even and three pairwise disjoint sets A, B, C of size m/2, put the lists A âª B, B âª C and A âª C on x1, x2, x3, and take t copies of each of these three lists on the y side, so l = 3t. No colour lies in all three lists, so every colouring of x1, x2, x3 uses two distinct colours Î± and Î²; some y-type contains both, and each y-type is one of the three lists, so every factor is at most mâ1. Hence P(G,L) â¤ mˆ3 (mâ1)ˆ(2t) (mâ2)ˆt while P(K_{3,l},m) â¥ m(mâ1)ˆl, and P_â < P once t ln((mâ1)/(mâ2)) > 2 ln m, that is for t > 2(mâ1) ln m. So Ï(K_{3,l}) > m for m of order l/log l. Thus âl is not a uniform law in the number s of vertices on the small side, and s = 2 is special. This note was checked by hand and is not formalized.

6. The Lean certificate

The development uses Lean 4.34.1 with mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612, in the namespace ListCol. Colours are Finset Nat. The definitions, quoted from the module ListColBase, are:

```
noncomputable def countL (G : SimpleGraph V) [DecidableRel G.Adj] (L : V â Finset Nat) : Nat :=
  Fintype.card (ListColorings G L)

noncomputable def chromaticPolynomial (G : SimpleGraph V) [DecidableRel G.Adj] (m : Nat) : Nat :=
  countL G fun _ => Finset.range m

noncomputable def listColorFunction (G : SimpleGraph V) [DecidableRel G.Adj] (m : Nat) : Nat :=
  sInf {n : Nat | â L : V â Finset Nat, (â v, (L v).card = m) â§ n = countL G L}

noncomputable def tau (G : SimpleGraph V) [DecidableRel G.Adj] : Nat :=
  sInf {k : Nat | G.chromaticNumber â¤ (k : ââ) â§
    â m : Nat, k â¤ m â listColorFunction G m = chromaticPolynomial G m}

def K (l : Nat) : SimpleGraph (Fin 2 â Fin l) :=
  completeBipartiteGraph (Fin 2) (Fin l)
```

The literal endpoint and the counting theorem, quoted from the modules ListColEndpoint and ListColMainIneq, are:

```
theorem tau_isTheta :
    (fun l : â => (tau (K l) : â)) =Î[atTop] fun l : â => Real.sqrt (l : â)
theorem tau_completeBipartite_isTheta
    (inst : â l : â, DecidableRel (completeBipartiteGraph (Fin 2) (Fin l)).Adj) :
    (fun l : â =>
       (@tau (Fin 2 â Fin l) _ _ (completeBipartiteGraph (Fin 2) (Fin l)) (inst l) : â))
     =Î[atTop] fun l : â => Real.sqrt (l : â)
theorem countL_ge {l m : â} (hm : 4 â¤ m) (hl : 2 * l â¤ (m - 1) ^ 2)
    (L : Fin 2 â Fin l â Finset â) (hL : â v, (L v).card = m) :
    m * (m - 1) ^ l + m * (m - 1) * (m - 2) ^ l â¤ countL (K l) L
```

In these quotations the blackboard letters for the naturals and the reals are printed as N and R, and the proof term after := by is elided; every other token, including the binders and the typeclass arguments, is as in the module.

The content is carried by main_real, whose proof splits into caseOne (k = 1) and caseTwo (k â¥ 2) and uses the per-list inequality perList, the tangent-line AM-GM exp_avg and the scalar inequality scalar_main. This is the route of

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Section 3, with Real.exp tangent lines in place of real powers. No containment hypothesis is used: only miss_lower enters, which says that any list of size m misses at least |U| â |S| colours of U.

An independent audit by a separate AI reviewer replayed the seventeen-module packet from a fresh extraction: every module compiled and passed the kernel checker, thirty-four runs in all, each exit 0, peak sampled memory 2.95 GiB. A source scan found no sorry, admit, added axiom, unsafe or native_decide, and the axiom print for the seven principal theorems returned exactly [propext, Classical.choice, Quot.sound]. An environment walk from those roots covered 27,763 constants with none missing; the only axioms in the closure are those three. Statement fidelity was checked against the source and against the literal statement of the question; semantic controls in ControlBridge and ControlValues checked the closed form of P(K_{2,l},m) against a brute-force count, the graph and its properness condition, the minimum over assignments, a small exact value of Ï, and the every-larger-m clause.

Three fidelity points should be stated plainly. Colours are Finset Nat; the source allows an arbitrary colour set, and the relabelling bridge from an arbitrary colour type is proved only in the auditor's control modules, not in the sealed packet, where it remains an informal remark. The every-larger-m clause is separated from equality at k only for k below the chromatic number, by a control comparing a threshold defined with equality at k alone with one defined for every m â¥ k; a separation with k â¥ Ï(K_{2,l}) would be a graph with Î½(G) < Ï(G), which Question 2 of [1] records as open. Finally, tau_completeBipartite_isTheta fixes the Fintype and DecidableEq instances of the vertex type, and independence from those instances is again a control-module result.

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7. Limits and disclosures

The prior-art search behind this draft was bounded: a small number of search strings, the papers [1], [2] and [3] read in full, [4] and [5] read only in part, and no citation index, no MathSciNet and no zbMATH; it therefore supports no clearance beyond what was read.

The journal version of [1] could not be read: J. Graph Theory 105(3) 386-397, DOI 10.1002/jgt.23045, was identified from a search listing only, and the DOI resolved with an HTTP 302 to a publisher page that answered 403. Whether the final text changes Conjecture 5 is unknown here. Allred and Mudrock, arXiv:2505.05662v1 (May 2025), was read only in its abstract and introduction and scanned for keywords; arXiv:2509.16733v2 (Kaul, Marciaga and Mudrock) was scanned at keyword level only; arXiv:2607.16810v1 was read only to page 6. None was found to bear on the threshold of K_{2,l}, but none was read in full.

The mathematical content is about a page of Bernoulli and AM-GM. Every tool in it appears in [2]: the containment reduction is Lemma 5, the k = 1 trade-off is Lemma 6, the block counts are Lemma 12, the block AM-GM is Lemma 13, and the joint-miss AM-GM is in the proof of Lemma 15. What is not in [2] is the uniform-in-m step: the block bound for A' Ã B' at an arbitrary number of shared colours, the reduction of every k â¥ 2 to k = 2, and the scalar inequality. The architecture is therefore close to that of Theorem 7 of [2] made uniform in m, and the proof adds nothing at small m: it covers only l â¤ 4 at m = 4 and l â¤ 8 at m = 5, well inside the ranges that Theorem 7 already settles. No worldwide novelty is claimed, and the significance of the result is for others to judge.

AI assistance is disclosed. The draft and the Lean development were produced with AI assistance; the author is Myeongjin Kang. No refereeing or publication is claimed.

References

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[2] Hemanshu Kaul, Akash Kumar, Andrew Liu, Jeffrey A. Mudrock, Patrick Rewers, Paul Shin, Michael Scott Tanahara and Khue To, "Bounding the List Color Function Threshold from Above", arXiv:2207.04831v2 (4 November 2022); Involve 16 (2023) 849-882, DOI 10.2140/involve.2023.16.849.

[3] Fengming Dong and Meiqiao Zhang, "An improved lower bound of P(G,L) - P(G,k) for k-assignments L", arXiv:2206.14536v4 (21 February 2023).

[4] Hemanshu Kaul, Jeffrey A. Mudrock and Armin Straub, "The unlabeled list color function of disconnected graphs", arXiv:2607.16810v1 (18 July 2026).

[5] Allred and Mudrock, Enumerative Chromatic Choosability, arXiv:2505.05662v1 (May 2025).

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