A symmetric kernel certificate for Hetyei's second−kind interval transform

DRAFT − prepared with AI assistance (OpenAI Codex). No human author or affiliation is asserted. This is an unpublished research draft for review, with no priority claim and no self−acceptance. MathDB 347230 is the sole principal target; 347231 appears only in a remark.

Abstract. On the degree−five part of the free rational algebra in a and b, the second−kind interval transform has a nonzero symmetric kernel element with twenty terms. Consequently its kernel is larger than the antisymmetric subspace predicted in Hetyei's Conjecture 5.14 (published Conjecture 39). Exact rational elimination gives kernel dimension 13, against antisymmetric dimension 12. The operator is interpreted using the cited Ehrenborg−Readdy mixing rule. We explain the source misprint, print the certificate, and distinguish the finite calculations from the supplied Lean proof and its reported strict replay.

1. Posets, words, and the intended question

Write V_n = Q<a,b>_n for the rational space with basis all length−n words, and V for their direct sum. Concatenation is multiplication; the empty word is 1. Reversal, extended linearly, is denoted by *. Put S_n = {u in V_n : u* = u} and A_n = {u in V_n : u* = −u}. The characteristic is zero, so V_n = S_n direct−sum A_n.

Let P be a finite graded poset with minimum 0−hat, maximum 1−hat, and rank n+1. For S contained in {1,...,n}, let f_S count chains of proper elements whose rank set is S. Hetyei [H, Section 2] defines the upsilon invariant and the ab−index by the following source formulas:

```
Upsilon_P(a,b) = sum_S f_S u_S,
(u_S)_i = b if i in S, and a otherwise;
Psi_P(a,b) = Upsilon_P(a-b,b).
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Let I(P) consist of the closed intervals of P, ordered by inclusion. Definition 5.1 of [H] calls the second−kind interval transform the "multiset of subposets of I(P)" formed, for each x in P, by "all elements [y,z] in I(P) containing [x,x]". For graded P these are the intervals [[x,x],[0−hat,1−hat]]. The total ab−index is their sum (Definition 5.4):

```
T(P) = sum_(x in P) Psi_[[x,x],[0-hat,1-hat]].
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The displayed definition in the source has u in the summation index but x in the summand; we use the consistent index x. Proposition 5.5 identifies each component with [0−hat,x]* x [x,1−hat], where the star on a poset means order duality. No Eulerian hypothesis is imposed here; the conjecture concerns all of V_n.

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2. The mixing operator and the source misprint

The coproduct of Ehrenborg–Readdy [ER1], also [H, Definition 4.6], deletes one letter at a time:

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Delta(w_1...w_m) = sum_(i=1)^m
       w_1...w_(i-1) tensor w_(i+1)...w_m.        (1)
```

Sweedler notation abbreviates this sum as sum_w w_(1) tensor w_(2). The following formulas reproduce [ER1, Definition 9.1], writing M_(r,s,k)(u,v) for their M_(r,s)(u,v,k). The index set is k >= 2, r,s in {1,2}, k congruent to r+s+1 modulo 2. Terms outside it are zero.

```
M_(1,2,2)(u,v) = u a v;  M_(2,1,2)(u,v) = v b u;
M_(1,s,k+1)(u,v) = sum_u u_(1) a M_(2,s,k)(u_(2),v);
M_(2,s,k+1)(u,v) = sum_v v_(1) b M_(1,s,k)(u,v_(2));
M(u,v) = sum_(r,s,k in the index set) M_(r,s,k)(u,v). (2)
```

These are bilinear rules. A recursion step deletes a letter from one argument, so M_(r,s,k)(u,v) = 0 when k > |u|+|v|+2. Thus (2) is a finite definition on words. Each resulting word has length |u|+|v|+1: each deletion in a recursive step is balanced by the inserted separator. Theorem 9.2 of [ER1] states that Psi_(P x Q) = M(Psi_P,Psi_Q). Hetyei cites precisely this theorem as equation (5.1).

Hetyei's Definition 5.6 (published Definition 31) instead prints M_(2,1,2)(u,v) = u b v. This is incompatible with its cited definition: [ER1] explicitly says an r=2 summand begins with a v–piece and an s=1 summand ends with a u–piece. It is also incompatible with the claimed product formula. Direct calculation gives

```
ER:       M(a,1) = aa + ab + ba;
printed:  M(a,1) = aa + 2ab.
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The product of a chain of rank two and a chain of rank one has ab–index aa+ab+ba. Thus this elementary test distinguishes the readings without assuming the kernel conjecture. The ER operator is symmetric in its two inputs, by the product identity on spanning poset indices (or the alternating–piece formula).

Theorem 5.7 (published Theorem 32) of [H] states the source formula

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T(P) = I_2(Psi_P), where
I_2(u) = u + u* + sum_u M(u_(1)*,u_(2)).          (3)
```

We use (3) with (2), the ER reading. It preserves degree. The printed codomain V_(n+1) in Conjecture 5.14 is therefore a slip, not a degree–raising definition. Root's supplied ruling adopts this reading. With the printed base case, I_2(ab–ba)=2ab–2ba, contradicting the source's Corollary 5.8; that failure is solely a misprint artefact and is not the result claimed here.

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3. The conjecture and an explicit certificate

Corollary 5.8 of [H] reads verbatim (with mathematical symbols transcribed): "If the homogeneous ab−polynomial u in Q<a,b>_n satisfies u* = −u then I_2(u) = 0." Indeed, applying (1) to a reversed word interchanges and reverses the two coproduct pieces. Symmetry of M then gives I_2(w*)=I_2(w), and hence A_n is contained in the kernel.

Conjecture 5.14 reads verbatim: "For each n >= 1, the kernel of I_2: Q<a,b>_n –> Q<a,b>_(n+1) is A_Q<a,b>_n." This is Conjecture 39 in the published version and the question recorded as MathDB 347230. Throughout, its intended kernel is the kernel of (3) restricted to V_n.

Theorem. Under the ER reading (2), ker(I_2 restricted to V_5) is not A_5. In fact its dimension is 13 whereas dim A_5 = 12. The following homogeneous polynomial is nonzero, symmetric, and killed by I_2:

```
u =  aaaba + aaabb − 2aabaa − aabba + aabbb
   + abaaa + abaab − abbaa + abbab − 2abbba
   − 2baaab + baaba − baabb + babba + babbb
   + bbaaa − bbaab − 2bbabb + bbbaa + bbbab.        (4)
```

Proof of the inequality. Every word in (4) has length five. Reversal pairs terms with the same coefficient, including the palindromes; thus u*=u. The coefficient of aaaba is 1, so u is not zero. Substitution of (4) into (1)−(3) gives I_2(u)=0. This last assertion is an exact finite computation, not a structural identity being assumed. Its checkable form is specified below and is independently certified by the supplied formal endpoint. If u were in A_5, symmetry and antisymmetry would imply 2u=0, contrary to its nonzero coefficient. This proves the conjecture false. The dimension assertion follows from the elimination described in Section 4.

A human−checkable route for the substitution is to enumerate the 32 output words lexicographically, a before b. For each of the 20 input words in (4), start with w+w*, then use its five deletion cuts and (2), whose index need not exceed six in this case. Record the resulting 32 coefficients, multiplied by the input coefficient. All column totals are zero. The dossier gives precisely this 20 by 32 signed table in certificate−contributions.tsv, with a totals row, and the word−by−word operator matrix.

Each input image has coefficient sum 64; the sum of the absolute input coefficients is 24. Consequently a fully expanded check with multiplicity has 1536 signed unit contributions. The unweighted product 20 times 64 is 1280, but is not the count for the signed certificate with coefficient multiplicities. The fresh standard−library script produces both the exact coefficient table and the checks u*=u and I_2(u)=0, using integer arithmetic for this part.

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4. Exact dimensions and the poset interpretation

The dimension assertions are computations over Q, separate from the certificate argument. For each n, order the words lexicographically, and form the square matrix whose column w is I_2(w). Rational Gauss–Jordan elimination, using the first available pivot in each column, gives the following results. The matrix, pivot choices, and reduced matrix are saved for each row of the table; no numerical tolerance or modular rank inference is used.

n	dim V_n	rank I_2	dim ker I_2	dim A_n	dim(ker cap S_n)
1	2	2	0	0	0
2	4	3	1	1	0
3	8	6	2	2	0
4	16	10	6	6	0
5	32	19	13	12	1
6	64	34	30	28	2
7	128	65	63	56	7

Reversal fixes exactly 2ˆceil(n/2) words. Every other reversal orbit contributes one antisymmetric basis vector w−w*. Therefore dim A_n = (2ˆn − 2ˆceil(n/2))/2. Since A_n is killed and V_n splits into symmetric and antisymmetric parts, the last column is the dimension of ker(I_2) intersect S_n. In particular, at degree five the extra symmetric direction is one−dimensional and is spanned by (4). The table records computations only; it asserts no general formula for larger n. Equality of the two dimensions at smaller degrees, together with the inclusion from Corollary 5.8, also verifies the conjectured equality there.

To pin the interpretation to the poset construction, consider D(n,S), with unique endpoints of ranks zero and n+1, two elements at every rank in S, and one at every other internal rank. Any two elements of different ranks are comparable in their rank order. For a rank set T, its flag entry is

```
f_T(D(n,S)) = 2^|T intersect S|.                (5)
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The matrix (5), indexed by all S,T, is a tensor power of the invertible matrix [[1,1],[1,2]]. Changing from flag entries to the ab−index is invertible as well: it is the substitution a −> a−b, b −> b. Hence these poset indices form a basis of V_n. This is an algebraic spanning proof, independent of rank tests.

The fresh script also enumerates the poset construction directly. For every x, it forms all intervals [i,j] with i <= x <= j, orders them by inclusion, assigns rank rho(j)−rho(i), enumerates all proper chains, and forms the flag ab−index. It sums over x and compares with (3). For every D(n,S) in degrees one through five, the two outputs agree. In degree five this compares all 32 basis indices and gives zero discrepancies; exact elimination also confirms their span rank is 32. The corresponding lower−degree spanning tests are retained in computations/results.json.

Thus the failure at degree five is a failure for the linear transform induced by the total ab−index of Hetyei's interval construction. The witness (4) is an element of the rational span of poset indices; we do not claim it is itself the index of a single poset. No universal poset−equivalence theorem is proved by this bounded test. It suffices to identify the entire degree−five operator independently of the misprinted base case, and agrees with the earlier independent review retained in the dossier.

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5. Companion remarks and formal correspondence

Remark on 347231. This is not a second principal result or a separate credit claim. Conjecture 5.15 (published Conjecture 40) asks for a generating set of S_n consisting of eigenvectors, obtained by compositions of the pyramid and lift operators. Put s=aab+baa. Direct use of (3) gives

```
I_2(s) = 2aaa + 8aab + 6aba + 4abb
                    + 6baa + 4bab + 2bba.          (6)
```

This is not symmetric, so S_3 is not invariant under I_2. A space spanned by eigenvectors is invariant, which already precludes the proposed generating set. The source lift is L(v)=(a−b)v+v(a−b). For p=aa+2ab+2ba+bb, one has I_2(p)=8p but

```
I_2(L(p)) − 8L(p) = −12aba − 12abb + 12baa + 12bab.
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This also disproves the eigenvalue−preservation claim of Lemma 5.11 (published Lemma 36). It is context, not a premise of the proof of the kernel refutation. The raw MathDB 347231 text instead uses the one−sided lift (a−b)v, the ER variant mentioned in [H]; its discrepancy with the source must be retained. The invariant−subspace obstruction (6) does not depend on a choice of lift. Neither this remark nor the false lemma is a separate formal endpoint in the packet.

The supplied Lean theorem is Hetyei347230.kernel_conjecture_false : not Hetyei347230.KernelConjecture. The definitions are explicit: Word := List Bool with false=a and true=b; ABPoly := Word −>0 Q (rational finitely supported functions). cop lists every deletion cut. Integer lists of word−coefficient pairs form the computable layer FSum, and toPoly maps these lists into ABPoly. MrsAux uses Nat.rec, with its base values and successor equations implementing (2). Mrs, rsVal, and inIndex impose the source index conventions.

The range theorem Mrs_eq_nil_of_large proves that indices above |u|+|v|+2 contribute nothing. Thus the finite list in Mw is an exact realization of the source sum, not an experimental cutoff. M is the bilinear extension. I2w is (3) on words and I2 its rational linear extension. star is the linear reversal map; deg n is the supported submodule on length−n words; anti n is deg n intersect ker(star+id). The literal proposition is

```
KernelConjecture := forall n : Nat, 1 <= n ->
                       deg n intersect ker I2 = anti n.
```

uList is (4), and uCert its toPoly image. uCert_deg, star_uCert, and uCert_ne_zero establish degree, symmetry, and nonvanishing. check_I2u checks that all output words have length five and all 32 possible coefficients vanish using decide +kernel. The lemmas I2_toPoly, toPoly_coeff and allWords_complete bridge this check to I2_uCert over Q. The endpoint then applies the conjectured equality at degree five and derives 2uCert=0. The packet proves the complete equality in degree two as kernel_equality_degree_two, a non−vacuity control; it does not formalize Corollary 5.8 in all degrees, the rank table, the poset tests, or the companion remark.

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6. Evidence, limits, and related work

The formal source−only ZIP is reused byte−identically. Its SHA−256 is

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c96ee05e7799bca2aa7a35961e931185425
ed44c8efd7160c73452d8a8d6054c
```

This lane safely extracted its members without executing them and checked the packet hash. No Lean, lake, leanchecker, SAT/ILP solver, network, or child agent was used here. The fresh manuscript arithmetic used capped python3 −I −B processes. The reproducibility files retain matrices, coefficient tables, exact raw MathDB lines, primary source copies and retrieval receipts, frozen−input hashes, and the original formal and review evidence.

The supplied root replay−result.json reports PASS for compilation and explicit−module leanchecker runs of Defs, Range, Data, Bridge, Compute, Endpoint, Control and Audit, using Lean 4.34.1 and the pinned mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612. Its endpoint axiom list is [propext, Classical.choice, Quot.sound]. The audits report no forbidden local or reachable declaration metadata, proof placeholders, added axioms, unsafe code, native_decide or implementation redirection. These are attributed replay reports, not runs performed by the manuscript lane. The archive includes the underlying receipts and logs.

The trust boundary includes the Lean kernel and its dependencies, fidelity of the mathematical encoding, and the explicit ER interpretation. Python elimination and poset enumeration are independently reproducible exact computations, but are not formal theorems in this packet. The counterexample endpoint does not depend on the computed rank or the literature screen. Root alone decides acceptance; this draft claims zero accepted credit.

Related work. Ehrenborg−Readdy [ER2] develop the Tchebyshev transforms and an eigenvector construction that motivates [H]. Hetyei's earlier Tchebyshev posets [H1] and triangulations [H2], and the generalizations with Nevo [HN], provide the combinatorial background. The kernel failure here concerns Hetyei's second−kind interval transform, and does not refute the ER transform results. Its significance is modest and concrete: a finite exact counterexample to a structural conjecture, with a symmetric kernel direction already in degree five.

The bounded prior−art screen supplied by discovery, independent review, and root found no relevant resolution. It checked cached primary papers, OpenAlex citing−work metadata and abstracts, later Hetyei−work metadata, GitHub searches, Lean Pool entries, and OpenAI/Omega registry paths. Root's dated registry refresh reports CLEAN, expressly as a path screen rather than a full−body proof search. Access limitations include Springer cookie redirection and unavailable scholarly indexes; citing−work coverage was OpenAlex−only. This lane made no new literature search. An open database label or a clean registry screen is not a novelty proof. No priority claim is made.

References (numbering in the text uses the arXiv version of [H]).

[H] G. Hetyei, The type B permutohedron and the poset of intervals as a Tchebyshev transform. Discrete Comput. Geom. 71 (2024), 918−944. DOI: 10.1007/s00454−023−00490−4. arXiv:2007.07362v2 (2020). Section 5 corresponds to published Section 6.

[ER1] R. Ehrenborg and M. Readdy, Coproducts and the cd−index. J. Algebraic Combin. 8 (1998), 273−299. Definition 9.1 and Theorem 9.2.

[ER2] R. Ehrenborg and M. Readdy, The Tchebyshev transforms of the first and second kind. Ann. Comb. 14 (2010), 211−244, especially Theorem 10.10.

[H1] G. Hetyei, Tchebyshev posets. Discrete Comput. Geom. 32 (2004), 493−520. [H2] G. Hetyei, Tchebyshev triangulations of stable simplicial complexes. J. Combin. Theory Ser. A 115 (2008), 569−592. [HN] G. Hetyei and E. Nevo, Generalized Tchebyshev triangulations. J. Combin. Theory Ser. A 137 (2016), 88−125. Bibliographic data for these background works follow the references of [H]; no unverified DOI is added.

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