The path P_9 is an IR-graph: a nine-vertex counterexample to the Mynhardt–Roux path conjecture

DRAFT manuscript, 5 November 2026. Not peer reviewed, not submitted, not accepted. Prepared with substantial AI assistance: the example was found by an AI-run computer search, and the proof write-up, recomputations and this text were produced by AI models (see the provenance statement at the end). No human author is listed; authorship and responsibility are to be assigned by the human operator. No priority claim is made.

Abstract. Mynhardt and Roux defined the IR-graph $G(\mathrm{IR})$ of a graph G: its vertices are the irredundant sets of G of maximum cardinality, and two of them are adjacent when one arises from the other by exchanging a single vertex for a neighbour. They proved that P_3, P_4, P_5, C_5, C_6 and C_7 are not IR-graphs, and conjectured that no path P_n with $n \geq 3$ and no cycle C_n with $n \geq 5$ is an IR-graph. We exhibit a graph G on 9 vertices with 20 edges, obtained from $K_{3,3,3}$ by deleting seven edges, such that $\mathrm{IR}(G) = 3$, G has exactly nine $\mathrm{IR}(G)$-sets, and $G(\mathrm{IR}) \cong P_9$. Hence the path conjecture is false. The proof is short and can be checked by hand from two small tables. It has also been formalised in Lean 4 with Mathlib, where the negation of the conjunction recorded as MathDB problem 350452 is a kernel-checked theorem. An exhaustive computer search over all graphs of order at most 9 shows that 9 is the smallest order of a graph whose IR-graph is a path of order at least 3, and that at order 9 exactly P_7, P_8 and P_9 occur. Whether P_6, or any cycle C_n with $n \geq 5$, is an IR-graph remains open.

mathdb:2611.00056v2 17 Nov 2026

1 The setting and the conjecture

Reconfiguration graphs record how feasible solutions of a problem can be transformed into one another by small steps. Fricke, Hedetniemi, Hedetniemi and Hutson [FHHH] introduced the γ-graph of a graph, whose vertices are its minimum dominating sets, with the "slide" adjacency described below. Mynhardt and Roux [MR-IG] began the study of the analogous graph for maximum irredundant sets. We use their definitions, which we quote from arXiv:1812.03382v3 (Sections 1.1 and 1.2). Graphs are finite and simple. $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood, and $N[D]$ is the union of the $N[v]$ for $v \in D$.

> "A D-private neighbour of $v \in D$ is a vertex v' that is dominated by v but by no vertex in $D - \{v\}$. … $\mathrm{PN}(v, D) = N[v] - N[D - \{v\}]$."

> "A set $D \subseteq V$ is irredundant if $\mathrm{PN}(v, D) \neq \emptyset$ for each $v \in D$ … the upper irredundance number $\mathrm{IR}(G)$ is the largest cardinality of an irredundant set of G. … an IR-set of G, sometimes also called an $\mathrm{IR}(G)$-set, is an irredundant set of cardinality $\mathrm{IR}(G)$."

> "… we define the IR-graph $G(\mathrm{IR})$ of G to be the graph whose vertex set consists of the $\mathrm{IR}(G)$-sets, i.e., the irredundant sets of G of maximum cardinality, where two $\mathrm{IR}(G)$-sets D and D' are adjacent if and only if there exist vertices $u \in D$ and $v \in D'$ such that $uv \in E(G)$ and $D' = (D - \{u\}) \cup \{v\}$."

> "To prove that a given graph H is an IR-graph, one needs to construct a graph G such that $G(\mathrm{IR}) \cong H$."

The paths P_n and cycles C_n have n vertices; the source treats P_3 as the star $K_{1,2}$.

Known results. Mynhardt and Roux [MR-IG] observe that $K_n(\mathrm{IR}) = K_n$. They show that Cartesian products of IR-graphs are IR-graphs, so every hypercube Q_n is one (the source of Q_n is nK_2), and in particular $C_4 = Q_2$ is. Every disconnected graph is the IR-graph of infinitely many graphs. An IR-graph of diameter 2 has an induced C_4, so the stars $K_{1,n}$ with $n \geq 2$ are not IR-graphs, and neither is any non-complete graph with a universal vertex. Part (ii) of their main theorem (labelled Thm_Main in the arXiv TeX source) states: "The cycles C_5, C_6, C_7 and the paths P_3, P_4, P_5 are not IR-graphs." The only connected IR-graphs of order 4 are K_4 and C_4. Among trees, the double star $S(2,2)$ is the unique smallest IR-tree of diameter 3, and the double spider $\mathrm{Sp}(1,1;1,2)$ is the unique smallest IR-tree of diameter 4. In [MR-IT] the same authors show that the IR-trees of diameter 3 are exactly the double stars $S(2n,2n)$. The survey [MN] summarises these results. The open-problems section of [MR-IG] reads:

> "A direct proof that P_5 is not an IR-graph is somewhat simpler than the proof of Theorem […], but not simple enough to easily generalize to a proof that longer paths and cycles are not IR-graphs. Nevertheless, we believe this to be true and state it as conjectures.
>
> **Conjecture 1.** P_n is not an IR-graph for each $n \geq 3$.

mathdb:2611.00056v2 17 Nov 2026

Conjecture 2. C_n is not an IR-graph for each $n \geq 5$."

Just before this, the authors add: "We conjecture that P_6 is not an IR-graph". The companion paper [MR-IT, Open Problems] restates the two conjectures as parts (i) and (ii) of one conjecture, and the survey [MN] states them as a single sentence. MathDB problem 350452, "Non-IR conjecture for paths and cycles", which cites [MR-IT], records exactly this conjunction: "The path P_n is not an IR-graph for each $n \geq 3$, and the cycle C_n is not an IR-graph for each $n \geq 5$."

Our result. We refute Conjecture 1, and hence the conjunction, with an explicit graph of order 9 (Theorem 1). We do not address Conjecture 2. As far as our bounded search found (Section 6), no realisation of a path of order at least 3 or of a cycle of order at least 5 as an IR-graph has been published.

2 The graph and the theorem

Let $V = \{0, 1, \ldots, 8\}$ with parts $A = \{0,1,2\}$, $B = \{3,4,5\}$ and $C = \{6,7,8\}$. Let G be the complete tripartite graph $K_{3,3,3}$ on these parts with the seven edges 05, 08, 14, 23, 26, 37 and 56 deleted. The deleted edges form the path 8–0–5–6–2–3–7 together with the edge 14. Thus G has the 20 edges

03 04 06 07 13 15 16 17 18 24 25 27 28 36 38 46 47 48 57 58.

Vertices 1, 4, 7 and 8 have degree 5 and the others have degree 4. The graph is connected, and a brute-force check of all 9! permutations shows that its automorphism group has order 2. For $S \subseteq V$ write $U(S) = V - N[S]$ for the set of vertices *not* dominated by S, with $U(\varnothing) = V$ and $U(v) = U(\{v\})$. Then $U(S)$ is the intersection of the sets $U(u)$, $u \in S$, and $U(u)$ is the neighbourhood of u in the complement of G. Table 1 lists the closed neighbourhoods and the sets $U(v)$.

v	deg	$N[v]$	$U(v) = V - N[v]$
0	4	{0,3,4,6,7}	{1,2,5,8}
1	5	{1,3,5,6,7,8}	{0,2,4}
2	4	{2,4,5,7,8}	{0,1,3,6}
3	4	{0,1,3,6,8}	{2,4,5,7}
4	5	{0,2,4,6,7,8}	{1,3,5}
5	4	{1,2,5,7,8}	{0,3,4,6}
6	4	{0,1,3,4,6}	{2,5,7,8}
7	5	{0,1,2,4,5,7}	{3,6,8}
8	5	{1,2,3,4,5,8}	{0,6,7}

Table 1. Closed neighbourhoods $N[v]$ and non-dominated sets $U(v) = V - N[v]$ in G.

Define the nine 3-sets

$$W_0 = \{0,1,2\},\ W_1 = \{0,2,7\},\ W_2 = \{0,4,7\},\ W_3 = \{4,6,7\},\ W_4 = \{6,7,8\},$$
$$W_5 = \{1,6,8\},\ W_6 = \{1,3,8\},\ W_7 = \{3,5,8\},\ W_8 = \{3,4,5\}.$$

Theorem 1. *IR(G) = 3, the IR(G)-sets of G are exactly $W_0, \ldots, W_8$, and two of them are adjacent in G(IR) if and only if their indices differ by one. Hence G(IR) ≅ P_9.*

Corollary 2. *P_9 is an IR-graph. Conjecture 1 of [MR-IG] (part (i) of the conjecture in [MR-IT]) is false, and so is the conjunction recorded as MathDB problem 350452.*

3 Proof of Theorem 1

Lemma 3 (heredity). *Every subset of an irredundant set is irredundant.* Indeed, if $D' \subseteq D$ and $v \in D'$, then $N[D' - \{v\}] \subseteq N[D - \{v\}]$, so $\mathrm{PN}(v, D) \subseteq \mathrm{PN}(v, D')$.

Lemma 4. *For every graph, every $D \subseteq V$ and every $v \in D$, $PN(v, D) = U(D - \{v\}) - U(D)$. In particular: (a) if $D - \{v\}$ dominates G, then v has no D-private neighbour; (b) if D dominates G, then $PN(v, D) = U(D - \{v\})$.* Indeed, $N[D] = N[v] \cup N[D - \{v\}]$, so $N[v] - N[D - \{v\}] = N[D] - N[D - \{v\}] = U(D - \{v\}) - U(D)$.

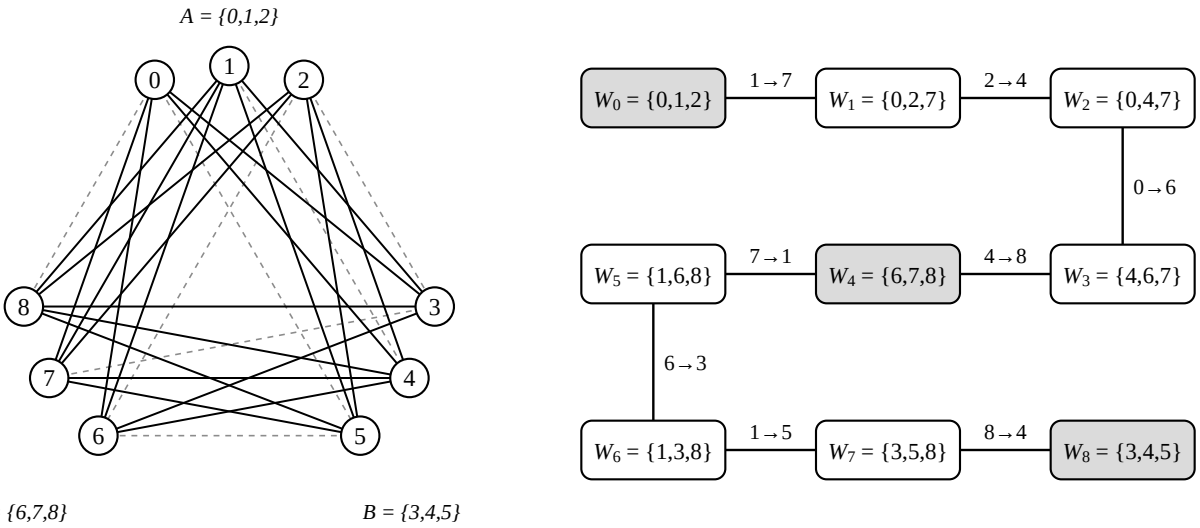


Figure 1. Left: the graph G (solid, 20 edges); the seven deleted edges of $K_{3,3,3}$ are dashed. Right: its IR-graph, the path $W_0 - W_1 - \dots - W_8$; the label $u \to v$ on an edge means that u is swapped out and its neighbour v is swapped in. Shaded sets are independent.

Step 1: pairs. Intersecting rows of Table 1 gives $U(P)$ for all 36 pairs P (Table 2). Exactly nine pairs dominate G, namely 05, 08, 14, 17, 23, 26, 37, 48 and 56: the seven deleted edges, which are non-edges of G, and the two edges 17 and 48 of G.

$x \setminus y$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	2	1	25	15	∅	258	8	∅
1		0	24	∅	04	2	∅	0
2			∅	13	036	∅	36	06
3				5	4	257	∅	7
4					3	5	3	∅
5						∅	36	06
6							8	7
7								6

Table 2. The set $U(\{x,y\})$ of vertices not dominated by $\{x,y\}$, written as a digit string; ∅ marks the nine dominating pairs.

Step 2: triples. Each of the 84 three-sets T falls into exactly one of three classes.

(a) T contains a dominating pair $T - \{v\}$. By Lemma 4(a), v has no private neighbour, so T is not irredundant. There are 54 such triples.

(b) T contains no dominating pair and does not dominate G. There are 21 such triples, listed in Table 3 together with a vertex $v \in T$ for which $U(T - \{v\}) = U(T)$. By Lemma 4, $\mathrm{PN}(v, T) = \varnothing$, so T is not irredundant.

(c) T dominates G and contains no dominating pair. By Lemma 4(b), $\mathrm{PN}(v, T) = U(T - \{v\}) \neq \varnothing$ for each $v \in T$, so T is irredundant. Checking the remaining triples against Table 2 shows that class (c) consists of exactly the nine sets $W_0, \dots, W_8$. Their private neighbourhoods, read off from Table 2, are listed in Table 4.

So G has exactly nine irredundant 3-sets. A direct count gives 1, 9, 36 and 9 irredundant sets of sizes 0, 1, 2 and 3; every pair is irredundant.

Step 3: no irredundant 4-set. Suppose that D is irredundant and $|D| = 4$. By Lemma 3 its four 3-subsets are irredundant, so by Step 2 they are four of the sets W_i, and any two of them share two vertices. Two of the sets W_i share two vertices only if their indices differ by one (inspect the 36 pairs). In particular, no three of them pairwise share two vertices, which is a contradiction. By Lemma 3 no irredundant set has more than three vertices, so $\mathrm{IR}(G) = 3$ and the $\mathrm{IR}(G)$-sets are exactly $W_0, \dots, W_8$.

For the record, here is an explicit witness for each of the 126 four-sets D. For 121 of them, some 3-subset $D - \{v\}$ dominates G (there are 63 dominating triples: classes (a) and (c)), and v has no private neighbour by Lemma 4(a).

mathdb:2611.00056v2 17 Nov 2026

T: $U(T)$, witness v	T: $U(T)$, witness v	T: $U(T)$, witness v
{0,1,3}: U = {2}, v = 3	{1,2,5}: U = {0}, v = 5	{2,5,7}: U = {3,6}, v = 2
{0,1,6}: U = {2}, v = 0	{1,2,8}: U = {0}, v = 2	{2,5,8}: U = {0,6}, v = 2
{0,2,4}: U = {1}, v = 4	{1,3,5}: U = {4}, v = 1	{2,7,8}: U = {6}, v = 2
{0,3,4}: U = {5}, v = 0	{1,3,6}: U = {2}, v = 3	{3,4,6}: U = {5}, v = 3
{0,3,6}: U = {2,5}, v = 6	{1,5,8}: U = {0}, v = 5	{3,6,8}: U = {7}, v = 3
{0,4,6}: U = {5}, v = 0	{2,4,5}: U = {3}, v = 2	{4,5,7}: U = {3}, v = 5
{0,6,7}: U = {8}, v = 0	{2,4,7}: U = {3}, v = 2	{5,7,8}: U = {6}, v = 5

Table 3. The 21 triples of class (b): T, then $U(T)$, then a vertex v with $U(T - \{v\}) = U(T)$, so that PN(v, T) = ∅.

The remaining five are exactly the four-element sets $U(w)$, namely $U(0)$ = {1,2,5,8}, $U(2)$ = {0,1,3,6}, $U(3)$ = {2,4,5,7}, $U(5)$ = {0,3,4,6} and $U(6)$ = {2,5,7,8}. In these the vertex 2, 0, 2, 0 or 2, respectively, has $U(D - \{v\}) = \{w\} = U(D)$, and so it has no private neighbour.

i	W_i	private neighbourhoods PN(v, W_i)	independent	to W_{i+1}
0	{0,1,2}	0: {0}; 1: {1}; 2: {2}	yes	1 → 7 (edge 17)
1	{0,2,7}	0: {3,6}; 2: {8}; 7: {1}	no	2 → 4 (edge 24)
2	{0,4,7}	0: {3}; 4: {8}; 7: {1,5}	no	0 → 6 (edge 06)
3	{4,6,7}	4: {8}; 6: {3}; 7: {5}	no	4 → 8 (edge 48)
4	{6,7,8}	6: {6}; 7: {7}; 8: {8}	yes	7 → 1 (edge 17)
5	{1,6,8}	1: {7}; 6: {0}; 8: {2}	no	6 → 3 (edge 36)
6	{1,3,8}	1: {7}; 3: {0}; 8: {2,4}	no	1 → 5 (edge 15)
7	{3,5,8}	3: {0,6}; 5: {7}; 8: {4}	no	8 → 4 (edge 48)
8	{3,4,5}	3: {3}; 4: {4}; 5: {5}	yes	—

Table 4. The nine IR(G)-sets with the private neighbourhoods PN(v, W_i) = $U(W_i - \{v\})$, and the exchange leading to W_{i+1}.

Step 4: adjacency. All IR(G)-sets have three vertices. So D and D' are adjacent in G(IR) if and only if $D - D' = \{u\}$ and $D' - D = \{v\}$ with $uv \in E(G)$. This relation is symmetric and irreflexive. Of the 36 pairs $\{W_i, W_j\}$, the 28 with $|i - j| \geq 2$ share at most one vertex, so they are not adjacent. For each of the eight pairs $\{W_i, W_{i+1}\}$, the exchanged vertices are 1 → 7, 2 → 4, 0 → 6, 4 → 8, 7 → 1, 6 → 3, 1 → 5 and 8 → 4, and 17, 24, 06, 48, 17, 36, 15 and 48 are edges of G. Hence G(IR) is the path $W_0W_1\ldots W_8$. ∎

Remark 5 (a problem from [MR-IG]). The authors also ask: “Prove or disprove: if G has an independent IR-set and the IR-graph H of G is connected and has order at least three, then H has maximum degree at least three.” In our graph, the independent sets W_0, W_4 and W_8 are the three parts A, C and B, and they are IR-sets. Yet $H = G$(IR) ≅ P_9 is connected and has maximum degree 2. So the same example answers this problem negatively. This is part of the same result and is not a separate claim.

Remark 6. The source's proof that C_6 and C_7 are not IR-graphs uses that these graphs have diameter 3. A corollary bounding the diameter from below then shows that a non-independent IR-set has exactly three vertices of positive degree. A lemma (Lem_Not_Path in the TeX source) then needs the distance between such a set and a certain “flip-set” to be 3, in order to force an induced C_4 or $S(2,2)$. Our IR-graph has diameter 8, so this argument gives no obstruction. The example says nothing about cycles.

4 Exhaustive search through order 9

The following are computations, not part of the proof of Theorem 1. All graphs were included, connected or not. For $n \leq 8$, the graphs on n vertices were generated up to isomorphism as one-vertex extensions of the graphs on $n - 1$ vertices, deduplicated by a canonical form. The canonical form uses colour refinement and individualisation, pruned only by transpositions of twin vertices. The class counts 1, 2, 4, 11, 34, 156, 1044 and 12346 agree with the known numbers of graphs [OEIS]. The IR-graph of each of these 13598 graphs was computed from the definitions. For order 9, all 256 one-vertex extensions of each of the 12346 graphs of order 8 were evaluated, 3160576 labelled

mathdb:2611.00056v2 17 Nov 2026

graphs in all. Every graph on 9 vertices arises this way up to isomorphism (delete any vertex). The results are in Table 5.

order n	what was evaluated	graphs evaluated	IR-graph P_m ($m \geq 3$) or C_m ($m \geq 5$)
1	all isomorphism classes	1	none
2	all isomorphism classes	2	none
3	all isomorphism classes	4	none
4	all isomorphism classes	11	none
5	all isomorphism classes	34	none
6	all isomorphism classes	156	none
7	all isomorphism classes	1,044	none
8	all isomorphism classes	12,346	none
9	all 256 one-vertex extensions of each order-8 class	3,160,576	21 labelled hits in 3 classes: P_7: 6 labelled, 22 edges; P_8: 10 labelled, 21 edges; P_9: 5 labelled, 20 edges (class of G)

Table 5. Exhaustive search for path IR-graphs P_m ($m \geq 3$) and cycle IR-graphs C_m ($m \geq 5$).

So, computationally: no graph of order at most 8 has P_m ($m \geq 3$) or C_m ($m \geq 5$) as its IR-graph. At order 9 there are exactly three isomorphism classes with a path IR-graph of order at least 3, giving P_7, P_8 and P_9; the last is the class of G. All three are connected and have IR = 3. No cycle C_m with $m \geq 5$, and no P_3, …, P_6, occurs through order 9. Thus 9 is the smallest order of a graph whose IR-graph is a path of order at least 3. Three implementations agree on these counts: the discovery search (connected graphs), the independent review (all graphs), and the fresh script of this manuscript (all graphs). They are not formally verified. While searching, the discovery lane also found path IR-graphs up to P_{14} among some graphs of orders 10 and 11. These observations are UNVERIFIED by review, were not exhaustive, and are not claimed. P_6, all cycles C_n with $n \geq 5$, and whether infinitely many paths are IR-graphs remain open.

5 Formal verification in Lean

The packet source.zip (SHA-256 953331d0…439ee; 8 modules General, GData, Four, Three, Finite, Endpoint, Control and Audit) was built with Lean 4.34.1 and Mathlib revision d13f23b7. It proves the endpoint

```
theorem IR350452.conjecture_false : ¬ IR350452.Conjecture
```

Table 6 records the statement correspondence.

source notion	Lean (namespace IR350452)
$w \in \mathrm{PN}(v, D)$	`IsPrivateNbr G D v w := (w = v ∨ G.Adj v w) ∧ ∀ u ∈ D, u ≠ v → ¬(w = u ∨ G.Adj u w)`
D irredundant	`Irredundant G D := ∀ v ∈ D, ∃ w, IsPrivateNbr G D v w`
IR(G)	`IR G := (univ.filter (Irredundant G)).sup card`, with `ir_upper` and `ir_attained`
IR(G)-set	`IRSet G D := Irredundant G D ∧ D.card = IR G`
$D \sim D'$ (slide)	`Swap G A B := ∃ u ∈ A, ∃ v ∈ B, G.Adj u v ∧ B = insert v (A.erase u)`
G(IR)	`IRGraph G : SimpleGraph {D // IsIRSet G D}`, `Adj := Swap`; symm by `swap_symm`, loopless by `swap_irrefl`
H is an IR-graph	`IsIRGraph H := ∃ (V : Type) (_ : Fintype V) (_ : DecidableEq V) (G : SimpleGraph V), Nonempty (H ≃g IRGraph G)`
the conjecture	`Conjecture := (∀ n, 3 ≤ n → ¬IsIRGraph (pathGraph n)) ∧ (∀ n, 5 ≤ n → ¬IsIRGraph (cycleGraph n))`
Theorem 1	`G9` (20-edge list on `Fin 9`), `ir_G9 : IR G9 = 3`, `pathIso : pathGraph 9 ≃g IRGraph G9`
Corollary 2	`conjecture_false : ¬ Conjecture`; control `nonvacuity : IsIRGraph (pathGraph 2)`

Table 6. Source notions and their Lean counterparts (namespace IR350452; definitions as printed by the root replay).

Adjacency is the literal swap relation. Nothing is symmetrised and no distinctness clause is added: symmetry on IR-sets follows from their equal cardinality (`swap_symm`), and irreflexivity from looplessness of G (`swap_irrefl`). `IsIRGraph` ranges over all finite simple graphs on vertex types in `Type`. This could only narrow the notion, so it cannot weaken a refutation. Mathlib's `pathGraph n` and `cycleGraph n` have n vertices. The finite steps are kernel `decide` proofs: all 126 four-sets are not irredundant, the 84 three-sets are irredundant exactly when they are one of

mathdb:2611.00056v2 17 Nov 2026

the nine walk sets, and the swap relation holds on the 81 ordered pairs of walk sets exactly for consecutive indices. Heredity, `irSet_iff_max` and `Equiv.ofBijective` then give `pathIso : pathGraph 9 ≃g IRGraph G9`. A non-vacuity control proves `nonvacuity : IsIRGraph (pathGraph 2)` (the IR-graph of K_2 is K_2).

Audits and replay. `#print axioms` reports only propext, Classical.choice and Quot.sound for both `conjecture_false` and `nonvacuity`. The Audit module checked all 77 local declarations for unsafe, partial, implemented_by and extern attributes and for local axioms. The source contains no sorry, admit or native_decide. The campaign root made a fresh strict replay from a safe extraction of the packet: each of the 8 modules was compiled with warnings as errors and checked by leanchecker (verdict PASS). The root also refreshed path screens of three public proof registries (verdict CLEAN; not a full-text search).

Trust limits. The author lane of this manuscript did not run Lean. It reused the packet byte-identically, checked its hash and performed a safe-extraction test. We rely on the Lean kernel, on leanchecker, on Mathlib's definitions of `pathGraph`, `cycleGraph` and graph isomorphism, and on the human judgement, documented above, that the definitions match the source.

6 Related work and prior-art screen

The γ-graph and its slide adjacency go back to Fricke et al. [FHHH], who showed that every tree is a γ-graph and conjectured that every graph is; Connelly, Hedetniemi and Hutson [CHH] proved this. Mynhardt and Teshima [MT] showed that every graph is a slide π-graph for several domination parameters π, including Γ, and mentioned the study of IR-graphs as an open problem (as reported in [MR-IG]). Irredundance was introduced by Cockayne, Hedetniemi and Miller [CHM]; see also the chapter [MR-ch]. Reconfiguration of dominating sets and colourings is surveyed in [MN]. In contrast with γ-graphs, not every graph is an IR-graph [MR-IG], and paths and cycles were among the first graphs tested.

Bounded screen (campaign lanes, 5 November 2026). All arXiv versions of [MR-IG] (v1–v3), [MR-IT] (v1, v2) and [MN] were read; none resolves or withdraws the conjectures. OpenAlex lists seven works citing the two journal articles and [MR-ch], and none concerns IR-graphs of paths or cycles. Three further OpenAlex queries returned HTTP 503. A few web searches, GitHub repository searches and path screens of the Lean Pool, OpenAI/math and Omega registries found no realisation of a path or a cycle as an IR-graph. The journal versions [MR-IG, MR-IT] (ScienceDirect) were not read. We cannot exclude that they, or works this screen missed, contain related material. We make no priority claim.

7 Significance and limitations

The contribution is modest: one explicit small graph that refutes a conjecture stated in [MR-IG] and restated in [MR-IT] and [MN], which its proposers supported with proofs for paths of order $n \le 5$. The proof is elementary. The example also answers negatively the problem quoted in Remark 5, which the authors hoped would help prove the conjectures. Only Conjecture 1 (part (i)) is refuted. Conjecture 2 on cycles is untouched, the status of P_6 is open, and we prove nothing about infinitely many paths. The minimality of order 9 and the census at order 9 rest on computation.

Provenance and AI assistance. The graph was found by a computer search run by an AI agent (Anthropic Claude Opus 5.5, campaign lane expanded-corpus-r209). The candidate was independently reviewed and re-searched by a second Claude lane (irpath-review-r1). The Lean packet was written by an OpenAI GPT-6.1 lane (irpath-formal-r1) and replayed by the campaign root. This manuscript, its tables and its fresh recomputations were produced by a Claude lane (irpath-manuscript-r1) with standard-library Python. Every number printed here was recomputed by scripts in the accompanying dossier. No human has yet checked this text.

References

[CHH] E. Connelly, S. T. Hedetniemi and K. R. Hutson, A note on γ-graphs, AKCE Int. J. Graphs Comb. 8(1) (2010) 23–31.

[CHM] E. J. Cockayne, S. T. Hedetniemi and D. J. Miller, Properties of hereditary hypergraphs and middle graphs, Canad. Math. Bull. 21 (1978) 461–468.

mathdb:2611.00056v2 17 Nov 2026

[FHHH] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi and K. R. Hutson, γ-graphs of graphs, Discuss. Math. Graph Theory 31 (2011) 517–531.

[MathDB] MathDB problem 350452, “Non-IR conjecture for paths and cycles”, https://mathdb.com/p/350452/non-ir-conjecture-for-paths-and-cycles (record as cached in the campaign corpus).

[MN] C. M. Mynhardt and S. Nasserasr, Reconfiguration of colourings and dominating sets in graphs: a survey, arXiv:2003.05956 (2020); published (per the arXiv source) in F. Chung et al. (eds.), 50 Years of Combinatorics, Graph Theory, and Computing, Chapman and Hall/CRC.

[MR-ch] C. M. Mynhardt and A. Roux, Irredundance, in: Developments in Mathematics, Springer, 2020, doi:10.1007/978-3-030-58892-2_6.

[MR-IG] C. M. Mynhardt and A. Roux, Irredundance graphs, arXiv:1812.03382 (v1 2018, v3 2021); Discrete Appl. Math. (2022), doi:10.1016/j.dam.2022.08.005 (journal version not read).

[MR-IT] C. M. Mynhardt and A. Roux, Irredundance trees of diameter 3, arXiv:2104.09004 (v2 2021); Discrete Math. (2022), doi:10.1016/j.disc.2022.113079 (journal version not read).

[MT] C. M. Mynhardt and L. Teshima, A note on some variations of the γ-graph, J. Combin. Math. Combin. Comput. 104 (2018) 217–230.

[OEIS] The On-Line Encyclopedia of Integer Sequences, A000088 (number of graphs on n unlabeled nodes).

[Lean] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, CADE-28 (2021); The mathlib Community, The Lean mathematical library, CPP 2020.

mathdb:2611.00056v2 17 Nov 2026