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A negative coefficient for every n ≥ 8 in the Alexandersson–Kebede cycle-restricted sum

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Abstract

We refute Conjecture 5.2.1 of Alexandersson and Getachew Kebede in its refereed version. For every n ≥ 8, restricting permutations to cycles of length at least 3 gives a signed polynomial whose normalized coefficient of $x_{\{1,n-4,n-2,n-1\}}\, y_{\{2,\ldots,n-5,n-3,n\}}$ is −1. A forced prefix reduces the coefficient to eight permutations, of which five survive the cycle restriction. We give the complete hand proof. A supplied Lean certificate proves the dual negation of the refereed and MathDB wordings at (n,k) = (9,3); the infinite family is not formalized. Fresh exact computations and a byte-preserved certificate accompany this draft.

1. Source setting and the positive derangement case

We use the commuting variables and one-based indexing of [1]. Write [n] = {1,…,n}, and let S_n act on [n]. The source defines an inversion as “a pair (i,j), for $1 \le i < j \le n$, such that $\pi(i) > \pi(j)$” and states: “The sign of π, sgn(π) is defined as $(-1)^{\mathrm{inv}(\pi)}$.” Its statistic definitions are quoted here with mathematical typography restored:

$\mathrm{EXCi}(g) := \{j \in [n] : g(j) > j\}$,
$\mathrm{EXCv}(g) := \{g(j) : j \in \mathrm{EXCi}(g)\}$,
$\mathrm{RLMi}(g) := \{i \in [n] : g(i) < g(j) \text{ for all } j \in \{i+1,\ldots,n\}\}$,
$\mathrm{RLMv}(g) := \{g(i) : i \in \mathrm{RLMi}(g)\}$,
$\mathrm{inv}(g) := |\{(i,j) : 1 \le i < j \le n \text{ such that } g(i) > g(j)\}|$.

The source says: “Whenever $S = \{s_1,\ldots,s_m\}$ is a finite set of positive integers, we shall let x_S denote the product $x_{s_1} x_{s_2} \cdots x_{s_m}$ of (commuting) variables. By definition, $x_\varnothing := 1$.” We use y_S analogously. These are sets of values, not positions; injectivity makes the MathDB use of “multisets” immaterial here.

Let D_n be the derangements. Theorem 1.0.1 of [1], refining the signed excedance result of Mantaci–Rakotondrajao [2], states for $n \ge 1$:

$$\sum_{\pi \in D_n} (-1)^{\mathrm{inv}(\pi)}\, x_{\mathrm{RLMv}(\pi)}\, y_{\mathrm{EXCv}(\pi)} = (-1)^{n-1} \sum_{j=1,\ldots,n-1} x_1 \cdots x_j\, y_{j+1} \cdots y_n.$$

Thus positivity after normalization is known when only fixed points are forbidden. Our obstruction concerns the next cycle-length restriction.

2. Which conjecture is being refuted?

In §5.2 of [1], cycle type is defined as “the integer partition $(\mu_1,\mu_2,\ldots,\mu_\ell)$ where the parts are the cycle lengths of π arranged in decreasing order.” In particular, fixed points contribute parts equal to 1. We transcribe the version-of-record conjecture verbatim, with the displayed formula set inline with the text:

“Conjecture 5.2.1. Let k be a fixed positive integer such that $2 \le k \le n$. Then

$$(-1)^{n-1} \sum_{\pi\in S_n,\ \min(\mathrm{type}(\pi))\ge k} (-1)^{\mathrm{inv}(\pi)}\, x_{\mathrm{RLMv}(\pi)}\, y_{\mathrm{EXCv}(\pi)},$$

where type(π), the cycle type of π, is an element in $\mathbb{N}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. The case k = 2 follows from Theorem 1.0.1 (the conjecture is not true for k = 1). For the case k = n, all permutations have the same sign, so the statement is trivial.”

The grammatical placement of “where type(π), the cycle type of π” is retained. Polynomial membership refers to the displayed sum. Define that normalized sum to be $F_{n,k}$, and denote the same sum without the factor $(-1)^{n-1}$ by $G_{n,k}$.

Version distinction. arXiv v3 and MathDB 350721 instead say: “Let k be a fixed positive integer such that $1 \le k \le n$. Then” followed by $G_{n,k}$, and “where type(π) is the cycle type of π, is an element in $\mathbb{N}[x_1,\ldots,x_n,y_1,\ldots,y_n]$.” There is no normalization factor. The next sentence says that k = 2 follows from the main theorem.

Versions v1 and v2 have the identical, malformed assertion: “Let k be a fixed positive integer. Then”

$$(-1)^n \sum_{\pi\in S_n} (-1)^{\ell(\mathrm{type}(\pi))} \left(\prod_{j\in \mathrm{RLMv}(\pi)} x_j\right)\left(\prod_{j\in \mathrm{EXCv}(\pi)} y_{\sigma(j)}\right)$$

“is an element in $\mathbb{N}[x_1,\ldots,x_n,y_1,\ldots,y_n]$.” Here k is absent from the sum, and the displayed σ is retained from the source. The following note asserts “$(-1)^{\ell(\mathrm{type}(\pi))-1} = (-1)^{\mathrm{inv}(\pi)}$” and refers to k = 1. We do not repair or use that assertion.

The k = 1 and even-n sign failures of the v3/MathDB statement are trivial wording defects and are not claimed as mathematical contributions. The supplied root ruling scores the substantive question against the refereed statement. Our odd-n witness refutes both later readings without exploiting either defect.

3. The negative-coefficient family

Theorem. For $n \ge 8$ put $R = \{1,n-4,n-2,n-1\}$ and $E = [n] \setminus R = \{2,\ldots,n-5\} \cup \{n-3,n\}$. Then

$$[x_R y_E]\, G_{n,3} = (-1)^n,$$
$$[x_R y_E]\, F_{n,3} = -1.$$

Consequently Conjecture 5.2.1 fails for k = 3 at every $n \ge 8$.

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4. Proof: forcing the complete list

A permutation contributes to the indicated monomial exactly when RLMv(π) = R and EXCv(π) = E. Since E and R are complements, each v ∈ E has pos(v) < v, and each v ∈ R has pos(v) ≥ v, where pos = π^{-1}.

The value in the last position is a right-to-left minimum. Right-to-left minimum values increase with their positions: if i < j are such positions, π(i) < π(j). Hence π(n) = max R = n−1. The value 2 must be at position 1. Inductively, for 2 ≤ v ≤ n−5, the requirement pos(v) < v and the already occupied positions force pos(v) = v−1. Thus π(i) = i+1 for 1 ≤ i ≤ n−6.

Set c = n−5. The remaining positions c,c+1,c+2,c+3,c+4 contain {1,c+1,c+2,c+3,c+5}; the final value is c+4. The excedance conditions force pos(c+2) ∈ {c,c+1}, pos(c+1) ≥ c+1, and pos(c+3) ∈ {c+3,c+4}. The RLM conditions force

pos(1) < pos(c+1) < pos(c+3).

If pos(c+3) = c+3, the value after it must be c+5, since it is a right-to-left minimum. Now c,c+1,c+2 contain 1,c+1,c+2. If c+2 occurs at c+1, their order is (1,c+2,c+1); if it occurs at c, their order is (c+2,1,c+1). These give B1 and B2.

If pos(c+3) = c+4, then the first four tail positions contain 1,c+1,c+2,c+5. If pos(c+2) = c, ordering 1 before c+1 leaves three orders for the other values, giving A1, A2, A3. If pos(c+2) = c+1, the condition pos(c+1) ≥ c+1 puts c+1 at c+2 or c+3. The former gives A4; the latter gives A5 or A6. These cases exhaust all possibilities.

Name	Tail at positions c,…,c+4	Cycle lengths	inv at n=9
A1	c+2, 1, c+1, c+5, c+3	n−3, 3	7
A2	c+2, 1, c+5, c+1, c+3	n	8
A3	c+2, c+5, 1, c+1, c+3	n−4, 4	9
A4	1, c+2, c+1, c+5, c+3	n−5, 3, 2	6
A5	1, c+2, c+5, c+1, c+3	n−5, 5	7
A6	c+5, c+2, 1, c+1, c+3	n	10
B1	1, c+2, c+1, c+3, c+5	n−5, 2, 2, 1	5
B2	c+2, 1, c+1, c+3, c+5	n−3, 2, 1	6

Conversely, every listed tail satisfies the required value sets. In each, 1 precedes c+1, which precedes c+3, followed by c+4 at the final position. Their later values are larger, so precisely these four values are RLMs. Every prefix value has the later value 1; the value c+2 has a later 1 or c+1; and c+5 has the later c+4. None of those values is an RLM. Direct position comparison gives exactly E as the excedance-value set.

The table therefore describes the whole coefficient fibre before imposing any restriction on cycles. No cancellation or cycle argument has been used to prune the list.

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5. Proof: cycles, signs and the boundary

To check the cycle column without extrapolating from experiments, let C denote the chain $1 \to 2 \to \cdots \to c$. In cycle notation the eight permutations decompose as follows; the notation (C,a,…) means the cycle (1,2,…,c,a,…).

A1: (C,c+2,c+1)(c+3,c+5,c+4);
A2: (C,c+2,c+5,c+4,c+3,c+1);
A3: (C,c+2)(c+1,c+5,c+4,c+3);
A4: (C)(c+1,c+2)(c+3,c+5,c+4);
A5: (C)(c+1,c+2,c+5,c+4,c+3);
A6: (C,c+5,c+4,c+3,c+1,c+2);
B1: (C)(c+1,c+2)(c+3)(c+4,c+5);
B2: (C,c+2,c+1)(c+3)(c+4,c+5).

All cycles have length at least 3 precisely for A1, A2, A3, A5 and A6, since $c = n-5 \geq 3$. A4 has a 2-cycle, while B1 and B2 have a fixed point. A cycle of length d has sign $(-1)^{d-1}$, as it is a product of d−1 transpositions. The usual inversion-parity sign therefore gives $(-1)^{\mathrm{inv}(\pi)} = (-1)^{n-\text{number of cycles}}$. The five surviving signs, in the above order, are $(-1)^n$ times 1,−1,1,1,−1. Their sum is

$$(-1)^n(1-1+1+1-1) = (-1)^n.$$

Multiplying by $(-1)^{n-1}$ yields −1, completing the proof. At n = 7 the forcing and the same list still apply, but A5 has a cycle of length 2 and is also excluded. The remaining four signs cancel. The target coefficient at this boundary is 0.

6. Fresh finite computations

The following are exact computations, not additional formal theorems. A new isolated Python script enumerated every permutation, computed the two value sets directly, decomposed all cycles including fixed points, and accumulated signs by monomial for every $2 \leq k \leq n$. Its cycle-parity signs were independently checked against literal inversion counts on every row of the family certificate for n = 7 through 16.

n	Permutations	k	Positive coefficients	Negative coefficients
8	40,320	3	256	2
9	362,880	3	707	15
10	3,628,800	3	1931	89
10	3,628,800	4	2997	3

For every $n \leq 7$ in the nonempty range $2 \leq k \leq n$, no negative coefficient occurred. At n = 8 and 9, exactly eight permutations have the target pair of value sets, agreeing with the hand proof. The n = 10 computation was divided into disjoint chunks by the first value, with coefficients combined before counting their signs. The dossier retains every chunk. It also contains the words, inversion counts, complete cycle types and target coefficients for n = 8 through 16, plus independent searches through all orders of the five free values.

Source typo. The polynomial labelled “for n = 3” after (23) of [1] is the n = 4 single-cycle sum. A fresh single-cycle calculation verifies this identity; this observation is only a transcription remark.

7. Formal correspondence and trust boundary

The supplied, unmodified Lean packet proves

AK350721.conjecture_refuted :
¬ AK350721.VoRConj ∧ ¬ AK350721.MathDBConj.

Both negations use n = 9, k = 3 and $x_{\{1,5,7,8\}}\, y_{\{2,3,4,6,9\}}$. Since the normalization is $(-1)^8 = 1$, its coefficient is −1 in both readings. This is an interior k and does not exploit the v3 sign defect.

Definitions. The source group S_n is encoded as Equiv.Perm (Fin n). The definition invCount counts ordered pairs i < j with π(j) < π(i); the sign is literally $(-1)^{\text{invCount}}$, with no unproved conversion from a sign homomorphism. The finite set excv consists of v with $\pi^{-1}(v) < v$. The set rlmv consists of v smaller than π(j) at every j later than $\pi^{-1}(v)$. These are precisely the source definitions after shifting both positions and values down by one.

The variable Sum.inl i represents x_{i+1}, and Sum.inr i represents y_{i+1}. The definition term is the product over the two value sets. cycleGe k π means every part of π.partition.parts is at least k, using Equiv.Perm.partition, which includes fixed points. rawSum is the signed sum of these terms, and F multiplies it by the constant $(-1)^{n-1}$.

InNat f is the literal existence of a polynomial p over ℕ whose image under MvPolynomial.map (Nat.castRingHom ℤ) equals f. VoRConj universally asserts InNat (F n k) for 2 ≤ k ≤ n; MathDBConj asserts InNat (rawSum n k) for 1 ≤ k ≤ n. The proved lemma InNat.coeff_nonneg supplies the direction needed for contradiction. Thus coefficient nonnegativity was not silently substituted for the source membership statement.

Finite characterization. The degreeSet bridge identifies a monomial with the pair of indicator sets. At n = 9, the zero-based sets are R = {0,4,6,7} and E = {1,2,3,5,8}. The formal proof forces inverse positions of 1,2,3 to be 0,1,2 and the value 7 to the last position. Bounded arithmetic classifies the five remaining inverse positions, giving the same eight permutations. It does not enumerate S_9. Concrete forward and inverse tables, value sets, inversion counts and complete partitions are kernel-checked. The surviving signs sum to −1. The packet also proves AK350721.nonvacuity_control : InNat (F 3 3).

What was checked here. We hash-verified the packet and safely extracted its members, rejecting unsafe paths, symbolic links and duplicate members. We ran no Lean, lake or leanchecker. The parent replay receipt reports separate successful builds and explicit-module leanchecker runs for every module, with Lean 4.34.1 and mathlib commit d13f23b723b8a846827a245b89c10fc7d3f11612. Its recorded axiom closure is only propext, Classical.choice and Quot.sound; its declaration metadata and source scans report no prohibited bridge. These are inherited audit results, not a fresh replay by this manuscript lane.

The formal certificate proves the finite witness and both negations. The all-n family above remains an ordinary mathematical proof. Trust additionally rests on statement correspondence, the Lean kernel and dependencies, and integrity of the supplied replay receipts. Computations and bounded literature searches do not establish a universal positivity theorem or priority.

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8. Related work, scope and reproducibility

Mantaci and Rakotondrajao [2] obtained the signed derangement count refined by excedance number. Alexandersson and Getachew Kebede [1] preserve excedance values and right-to-left minima in their involution, yielding the multivariate k = 2 identity. Pei and Zeng [3] study signed right-to-left minima and excedances on derangements. The cached full TeX discusses the open coefficient questions about $d_{n,k}$, which are different from the restriction on minimum cycle length considered here.

Ji and Zhang [4], identified in the supplied citing-work metadata, concern signed Mahonian polynomials on derangements in classical Weyl groups. Their primary full text was not available in the supplied evidence. We therefore do not claim a complete theorem-level exclusion of that work. The relationship recorded here is topical and bibliographic, rather than a claim that all of its results were checked.

The supplied review records a bounded prior-art screen of the source versions, nearby cached papers, OpenAlex citing records, public searches and GitHub repository searches. It found no resolution of this cycle-length conjecture through those channels. The parent registry refresh reports CLEAN for path and listing screens of OpenAI/math, Omega and Lean Pool at the heads recorded in its receipt. This is explicitly not a full-body proof search. Bibliographic databases or publisher bodies unavailable in the review could contain missed work. This manuscript lane used only frozen local inputs and makes no new live-source or exhaustive novelty claim.

The significance is modest: an exact persistent obstruction to a multivariate extension just beyond the established derangement identity. It does not diminish the k = 2 theorem, classify all negative monomials, or settle positivity at other restrictions. The finite computations show failure at k = 4 as well; they do not justify extrapolation to arbitrary k.

Dossier. The accompanying archive contains this editable manuscript source and PDF, the full hand proof, the exact raw MathDB line and hash, cached source and review copies with provenance receipts, the byte-identical Lean source ZIP, and a separate safe-extraction receipt. Fresh isolated scripts reproduce all displayed experimental counts and the finite eight-permutation tables. Before-and-after SHA-256 manifests bind the frozen inputs. Rendering images and a page-by-page QA receipt document the final PDF inspection. REPRODUCE.md distinguishes authorized Python reproduction from the inherited Lean replay procedure.

Numerical and rendering drivers impose the requested resource caps. A failed rendering setup import created bytecode caches outside the lane in the installed runtime; the dossier records this process violation. Bytecode writes are now explicitly disabled. No network, solver, child agent or shared-ledger change was used. Acceptance remains solely with the campaign root.

References

[1] P. Alexandersson and F. Getachew Kebede, *An involution on derangements preserving excedances and right-to-left minima*, Australasian Journal of Combinatorics **86**(3) (2023), 387–413. Version of record, §5.2, Conjecture 5.2.1; arXiv:2105.08455, versions v1–v3. Cached primary PDF and TeX are included.

[2] R. Mantaci and F. Rakotondrajao, *Exceedingly deranging*, Advances in Applied Mathematics **30** (2003), 177–188. Bibliographic data as cited in the cached primary sources.

[3] Y. Pei and J. Zeng, *Counting derangements with signed right-to-left minima and excedances*, Advances in Applied Mathematics **152** (2023), article 102599; arXiv:2206.11236. DOI: 10.1016/j.aam.2023.102599 (verified in the supplied citing metadata).

[4] K. Q. Ji and D. T. X. Zhang, *Signed Mahonian polynomials on derangements in classical Weyl groups*, European Journal of Combinatorics (2024), article 104083. DOI: 10.1016/j.ejc.2024.104083 (verified in the supplied citing metadata). No volume is inferred from that metadata.

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