An 88-vertex totient realization of the four-stage contracted nanostar

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Abstract. We give an explicit positive-integer labeling of the four-stage nanostar tree as an iterated Euler-totient graph. Its orbit graph has 88 vertices and centre 1024. This refutes the final nonexistence conjecture of Ghanbari and Alikhani at stage 4, under the contracted-hexagon interpretation of the nanostar recursion. That interpretation is supported by the published figure and the cited Deutsch-Klavzar construction; it is not an explicit contraction definition in the source. A complete arithmetic table, independent finite verification, and a byte-identical previously checked Lean packet accompany this draft. The contribution is a modest exact counterexample found by routine inverse-totient backtracking.

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1. Definitions and the source conjecture

For a positive integer n let phi(n) be Euler's totient, with phi(1) = 1. Define phi^0(n) = n and phi^(j+1)(n) = phi(phi^j(n)). For a set A of positive integers put A_phi = {phi^j(a) : a in A, j >= 0}. In particular, iteration zero is included, and 1 belongs to every nonempty orbit set. Let G_phi(A) have vertex set A_phi and unordered edges {n, phi(n)} with n in A_phi and n != phi(n). We omit the loop at 1 and forget direction, in accordance with the source's simple-graph convention and explicit undirected remark [1, p. 265].

A graph H is a G_phi-graph when it equals G_phi(A) after labeling its vertices, equivalently when it is isomorphic to G_phi(A). This explains the source's notation H = G_phi(A). Although the source allows arbitrary sets A, a finite realization has A finite because A is contained in A_phi. Thus finite positive witnesses suffice here.

The published final conjecture [1, p. 271], quoted verbatim with mathematical notation rendered as G_phi, is: *“There is no nanostar molecule which has grown at least four stages, as a G_phi-graph.”* The preprint uses the title *A graph related to Euler phi function*; the published title inserts “the.” The MathDB record 349177 paraphrases “dendrimer nanostar”; the exact frozen raw line and its hash are in the dossier.

Immediately before the conjecture, Theorem 11 states: *“The graph D_2 is a G_phi-graph.”* Its proof gives A = {3, 256, 376, 384, 564}. We reproduce and check this control, rather than infer impossibility at stage 3 from the conjecture's stage threshold. The preprint numbers the corresponding theorem and conjecture 3.4 and 3.5.

2. Which nanostar tree is meant?

Deutsch and Klavzar [2, Section 4.3] define a rooted graph F using two hexagons and endpoint edges, G_1 using a hexagon and a pendant root edge, G_k as a bouquet of G_(k-1), G_(k-1), and F at their roots, and D_k as a bouquet of three G_k. Here bouquet means identification of the specified roots; G_k is rooted at its remaining unique leaf. We interpret the tree structure by contracting each hexagon to one vertex, retaining junction vertices and intervening edges. Published Figure 10 [1, p. 271] and Figure 12 [2, p. 639] have the same contracted skeleton. In the latter the circles represent hexagons. Frozen page images permit direct comparison.

This contraction is our stated interpretation, not a sentence attributed to [1]. A cyclic molecular reading cannot give the intended G_phi-graph: nonempty positive-integer orbit graphs are trees. Indeed phi(n) < n for n > 1, and every vertex reaches 1. In a purported cycle its largest label would have two smaller neighbors, but only its single totient parent can be smaller. In particular, keeping molecular hexagons would already contradict Theorem 11. The distinct D_3[n] family in [3] is not silently substituted for the D_k recursion.

After contraction, F is a rooted path of length 3 and G_1 is a rooted edge. Removing the common central root gives the following recursive arms: arm(0) is one vertex; arm(j+1) starts with two unary path vertices, followed by a branching vertex carrying two disjoint copies of arm(j). For k >= 1, D_k consists of a new centre joined to the roots of three copies of arm(k-1). This is precisely the family used below and in the formal definitions.

If a_j is the order of arm(j), then a_0 = 1 and a_(j+1) = 3 + 2a_j, so a_j = 4*2^j - 3. Hence |D_k| = 1 + 3a_(k-1) = 6*2^k - 8. In D_4 branching occurs at distances 3, 6, and 9 from the centre; leaves occur at distance 10. The distance-layer profile is 1, 3, 3, 3, 6, 6, 6, 12, 12, 12, 24.

3. Explicit refutation and verification

Theorem. Under this contracted-hexagon interpretation, D_4 is a G_phi-graph. Consequently the assertion that no D_k with k >= 4 is realizable is false.

Take the following 23 generators A (all positive):

{3, 167, 334, 19577, 31657, 39154, 63314, 702085, 983055, 1048576, 1123336, 1975461, 2370559, 2633948, 3352289, 3561823, 4182461, 4741118, 5019445, 5866457, 8031112, 8364922, 11732914}.

Proof. Tables 1a and 1b give a set V of distinct labels and every totient value, computed by phi(n) = product of p^(e-1)(p-1) over the prime powers p^e in n. Each factorization is prime-certified by trial division. Every printed parent is in V; phi(1) = 1 is the only omitted loop. Following these parents from the listed generators visits exactly V. Thus V = A_phi, not merely a closed superset, and the table determines every edge of G_phi(A).

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Table 1a. Complete labeled orbit tree

Parent is phi(n), oriented toward 1; layer is distance from centre 1024, not orbit depth. At n = 1 the parent entry 1 denotes a suppressed loop. Products use prime bases and integer exponents.

Label n	Prime factorization	Parent phi(n)	Layer
1	1	1	10
2	2	1	9
3	3	2	10
4	2^2	2	8
8	2^3	4	7
16	2^4	8	6
32	2^5	16	5
40	2^3 * 5	16	7
64	2^6	32	4
82	2 * 41	40	8
128	2^7	64	3
166	2 * 83	82	9
167	167	166	10
256	2^8	128	2
320	2^6 * 5	128	4
334	2 * 167	166	10
512	2^9	256	1
704	2^6 * 11	320	5
1024	2^10	512	0
2048	2^11	1024	1
2208	2^5 * 3 * 23	704	6
3072	2^10 * 3	1024	1
4096	2^12	2048	2
4888	2^3 * 13 * 47	2208	7
5264	2^4 * 7 * 47	2208	7
8192	2^13	4096	3
9216	2^10 * 3^2	3072	2
9784	2^3 * 1223	4888	8
10544	2^4 * 659	5264	8
16384	2^14	8192	4
17408	2^10 * 17	8192	4
19576	2^3 * 2447	9784	9
19577	19577	19576	10
27648	2^10 * 3^3	9216	3
31656	2^3 * 3 * 1319	10544	9
31657	31657	31656	10
32768	2^15	16384	5
39154	2 * 19577	19576	10
52608	2^7 * 3 * 137	17408	5
63314	2 * 31657	31656	10
65536	2^16	32768	6
69760	2^7 * 5 * 109	27648	4
82944	2^10 * 3^4	27648	4
131072	2^17	65536	7

The table is sorted by label. Rows on the following page complete the same orbit set; no edges other than the listed non-loop parent edges are present.

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Table 1b. Complete labeled orbit tree (continued)

Parent is phi(n), oriented toward 1; layer is distance from centre 1024, not orbit depth. At n = 1 the parent entry 1 denotes a suppressed loop. Products use prime bases and integer exponents.

Label n	Prime factorization	Parent phi(n)	Layer
131680	2^5 * 5 * 823	52608	6
139264	2^13 * 17	65536	7
166912	2^10 * 163	82944	5
209472	2^6 * 3 * 1091	69760	5
262144	2^18	131072	8
280576	2^11 * 137	139264	8
329240	2^3 * 5 * 8231	131680	7
334336	2^9 * 653	166912	6
395088	2^4 * 3 * 8231	131680	7
419008	2^6 * 6547	209472	6
524288	2^19	262144	9
561664	2^9 * 1097	280576	9
658484	2^2 * 164621	329240	8
669184	2^9 * 1307	334336	7
702085	5 * 140417	561664	10
790184	2^3 * 98773	395088	8
836480	2^7 * 5 * 1307	334336	7
838048	2^5 * 26189	419008	7
983055	3 * 5 * 65537	524288	10
1048576	2^20	524288	10
1123336	2^3 * 140417	561664	10
1257072	2^4 * 3 * 26189	419008	7
1316972	2^2 * 329243	658484	9
1672976	2^4 * 104561	836480	8
1676128	2^5 * 52379	838048	8
1975461	3 * 658487	1316972	10
2007744	2^6 * 3 * 10457	669184	8
2370558	2 * 3 * 395093	790184	9
2370559	2370559	2370558	10
2633948	2^2 * 658487	1316972	10
2933224	2^3 * 7 * 52379	1257072	8
3352288	2^5 * 104759	1676128	9
3352289	3352289	3352288	10
3561823	17 * 209519	3352288	10
4015552	2^6 * 62743	2007744	9
4182460	2^2 * 5 * 209123	1672976	9
4182461	4182461	4182460	10
4741118	2 * 2370559	2370558	10
5019445	5 * 1003889	4015552	10
5866456	2^3 * 733307	2933224	9
5866457	5866457	5866456	10
8031112	2^3 * 1003889	4015552	10
8364922	2 * 4182461	4182460	10
11732914	2 * 5866457	5866456	10

Proof, continued. There are 88 vertices and 87 edges. The independent checker constructs the recursive D_4 separately, matches rooted canonical tree strings, and emits a bijection preserving all edges. It also compares every ordered pair against the actual Lean preorder labels. The three rooted components at 1024 each have the arm(3) shape, including all nonedges. This proves the required isomorphism; matching the layer profile alone would not suffice. The largest label is 11732914. A single stage-4 realization negates the universal conjecture. QED.

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4. Formal statement, bridge, and audit scope

The reused source packet formalizes R162.Tree with leaf, unary, binary, and ternary constructors. R162.size and R162.edges give disjoint preorder-indexed vertices and edges; treeGraph is the associated loopless SimpleGraph. R162.arm follows the recurrence in Section 2; nanostar k is ternary (arm (k-1)) (arm (k-1)) (arm (k-1)). Only k >= 4 is quantified in the conjecture, so natural-number subtraction at zero is immaterial.

R162.orbit A = {m : there exist a in A and j in N with (Nat.totient^[j]) a = m}. The adjacency of phiGraph A is u != v and (phi(u) = v or phi(v) = u). R162.Realizable t asserts existence of a Finset of positive natural numbers A and a graph isomorphism treeGraph t to phiGraph A. The exact proposition is:

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Conjecture349177 := forall k : N, 4 <= k ->
not Realizable (nanostar k)
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The bridge realizes_of_label takes an injective positive labeling, closure under totient, and an adjacency equivalence for every vertex pair. It uses all labels as the formal generator set. Induction on the iteration index proves that the orbit is exactly the label image; injectivity and surjectivity yield the graph isomorphism. This differs from the smaller 23-generator set only as a presentation of the same orbit graph.

Definitions supplies the general bridge; Arithmetic proves the totient identities; Certificate supplies the labels and parent indices. EdgeRows0 through EdgeRows7 check all adjacency rows, including nonedges. Endpoint constructs Realizable (nanostar 4) and concludes:

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R162.not_conjecture349177 :
not R162.Conjecture349177
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The twelve proof modules are reused byte-identically in proof-source.zip, SHA-256: 5aa1150cb685221b2f9dbfbab4cd9eff9f503d917add10b62a491a1e19d81fac.

The discovery and independent-review receipts report strict Lean 4.34.1 builds and explicit-module leanchecker checks, with an inventory of 335 local declarations. The only transitive axioms reported are propext, Classical.choice, and Quot.sound. No Lean, lake, leanchecker, or solver was run in this manuscript task; the fresh work here is independent Python arithmetic and graph verification plus archive validation.

The reviewer initially held the packet under an all-local-metadata rule: four compiler-generated partial runtime helpers exist, named R162.edges._unsafe_rec, R162.size._unsafe_rec, R162.arm._unsafe_rec, and R162.instDecidableEqTree.decEq._unsafe_rec. The independent dependency walk reports 271 safe, non-partial local endpoint dependencies and reaches none of these helpers. The root ruling resolves this as a qualified pass, invoking the ASEP 368807 precedent for unreachable compiler helpers; it does not pretend the four flags are absent.

Trust limits: the prior replays trust prebuilt mathlib/package oleans without a complete upstream rebuild or source-to-olean attestation; leanchecker uses the same Lean kernel foundation. Resource monitoring samples RSS and cannot exclude every transient peak. Root fresh replay and registry-refresh status are recorded separately in the final dossier receipt; neither is inferred from older logs.

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5. Controls, significance, and literature limits

The fresh checker verifies the source's D_2 witness {3, 256, 376, 384, 564}: 16 vertices, 15 edges, centre 16, maximum label 564, layers 1, 3, 3, 3, 6. The earlier discovery independent-control file has maximum 334 for its own different stage-2 witness. These are compatible, not conflicting computations of the same input.

As supporting data, the stage-3 generators are {3, 167, 334, 13107, 16384, 16449, 19577, 21932, 31657, 39154, 63314}. Their orbit gives 40 vertices, 39 edges, centre 128, maximum label 63314, and layers 1, 3, 3, 3, 6, 6, 6, 12. This independently verified control is not needed for the refutation and is not a separate contribution. The source neither proves nor excludes stage 3.

Not every tree is realizable. In K_(1,5), the vertex 1 would be a leaf and its neighbor would be 2: among positive integers other than 1 only 2 has totient 1. But phi(x) = 2 has exactly the solutions 3, 4, 6. To see exhaustion, every prime divisor p of x has p-1 dividing 2, so x = 2^a 3^b; the totient product then gives just those cases. Thus 2 has at most four neighbors, including 1, which excludes K_(1,5), as in [1, Theorem 5]. Large fibers at unspecified targets do not prescribe a compatible iterated tree rooted at 1.

The significance is modest: routine inverse-totient backtracking found an explicit counterexample to a published nonexistence conjecture. No general realization method, minimality, uniqueness, higher-stage theorem, or lower bound on search difficulty is claimed. Theorem 11 gives a single D_2 labeling, not an algorithm for arbitrary stages. The source's caterpillar and alkane constructions do not directly label this branching pattern.

The retained bounded literature review inspected the primary source, its published successor, the accessible chemical references, and the primary statements of cached totient-asymptotic and weighted-dilation/fiber manuscripts. The latter concern totient counts or large fibers, not the prescribed finite orbit geometry. Registry and title screens found no matching refutation in their stated scope; they do not clear all proof bodies, private work, or deleted history. No new network searches were made for this draft and no priority claim is made.

Specific unread items remain: *A graph related to the Tau function*, DOI 10.1063/5.0315935, is known only through retained metadata/abstract evidence; its full text was inaccessible. Rashid et al. [4] was also unread after a retained access denial. Neither is treated as affirmative evidence of absence of prior work. The root ruling gives qualified passes on fidelity and modest significance, with zero credit pending root acceptance. Campaign context is 41/60 new accepted results beyond 39 previous results; this draft assigns no additional credit.

References

[1] Nima Ghanbari and Saeid Alikhani. *A graph related to the Euler phi function.* The Mathematical Gazette (2023), pp. 263-272. DOI: 10.1017/mag.2023.57 (verified in the retained publisher PDF). Preprint: *A graph related to Euler phi function*, arXiv:2012.12492v1 (2020).

[2] Emeric Deutsch and Sandi Klavzar. *Computing the Hosoya Polynomial of Graphs from Primary Subgraphs.* MATCH Commun. Math. Comput. Chem. 70 (2013), 627-644, Section 4.3, Figures 9-12.

[3] S. Alikhani and M. A. Iranmanesh. *Hosoya polynomial of dendrimer nanostar D_3[n]*. MATCH Commun. Math. Comput. Chem. 71 (2014), 395-405.

[4] M. A. Rashid, S. Ahmad, M. Cancan, and M. H. Muhammad. *Topological properties of nanostar dendrimer and smart polymer*. J. Chem. 2020, Article 1280632. Bibliographic details from [1]; full text unread. No DOI is asserted here.

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