# Equitable partitions of finite rectangular grids are symmetry orbits

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## Abstract

For every pair of positive integers m and n, every equitable partition of the finite grid $P_m \square P_n$ is an orbit partition for a subgroup of its graph automorphism group. We prove this by propagating colour-preserving symmetries from the boundary, using multiset cancellation and the first arrival of side walks at corners. We also prove the two lattice assertions in Neuberger, Sieben and Swift's Conjecture 5.6: the equitable-partition lattice is isomorphic to the full automorphism subgroup lattice precisely when there is a free vertex, and the only exceptions are the squares of sides 2 and 3. The supplied Lean endpoint is unconditional. Root reports a separate successful replay; acceptance remains a separate decision. The substantive contribution is modest and concerns the first assertion for $m,n \geq 2$; paths are already covered by reported tree-network results.

## 1. Source, definitions and theorem

Write $[m] = \{1,\ldots,m\}$. The path $P_m$ has edges between consecutive integers; its box (Cartesian) product with $P_n$ has vertex set $V = [m] \times [n]$, with adjacency $|i-i'| + |j-j'| = 1$. There are no periodic boundary edges. Let $\mathrm{Aut}(G)$ be the full group of adjacency-preserving bijections of V. Its natural action is evaluation, $g \cdot v = g(v)$. For $H \leq \mathrm{Aut}(G)$, its orbit partition is $\{Hv : v \in V\}$.

Definition 5.1 of [1] calls a partition $\pi$ equitable when $|N(u) \cap C| = |N(v) \cap C|$ for every cell C and every pair u,v in one cell of $\pi$. Write $\mathrm{EqPart}(G)$ for these partitions. Order partitions by refinement: $\pi \leq \sigma$ means every $\pi$-cell lies inside a $\sigma$-cell. Order $\mathrm{Sub}(\mathrm{Aut}(G))$ by inclusion. A free vertex x has trivial stabilizer in the full group: $gx = x$ implies $g = 1$.

Conjecture 5.6 [1], verbatim (with the box-product symbol typeset): “The equitable partitions of $P_m \square P_n$ are the orbit partitions of natural actions of the subgroups of the automorphism group of the graph. The lattice of equitable partitions is isomorphic to the lattice of subgroups exactly when there is a vertex with trivial stabilizer subgroup. This happens unless $2 \leq m = n \leq 3$.” The source reports computations through $m = n = 20$ and some larger graphs. These are evidence, not a proof.

Theorem 1. For all integers $m,n \geq 1$: (i) $\pi$ is equitable if and only if $\pi$ is the orbit partition of some $H \leq \mathrm{Aut}(G)$; (ii) $\mathrm{EqPart}(G)$ and $\mathrm{Sub}(\mathrm{Aut}(G))$ are isomorphic lattices if and only if G has a free vertex; (iii) G has a free vertex if and only if (m,n) is neither (2,2) nor (3,3). In the free cases the natural map $\Phi(H) = \{Hv\}$ itself is an order and lattice isomorphism. All lattice assertions use the refinement convention above.

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## 2. Colours, determinism and positive layers

For a fixed equitable partition, let $\chi(v)$ be the cell of v. There is a multiset $M(c)$ of neighbour colours for every cell c, satisfying $M(\chi(v)) = [\![\chi(w) : w \in N(v)]\!]$. Equality of these multisets is exactly Definition 5.1: the multiplicity of a colour C is $|N(v) \cap C|$. Repeated colours are retained throughout. If g is an automorphism, $\chi \circ g$ satisfies the same rule M, because g bijects $N(v)$ onto $N(gv)$.

Let $R(i,j) = (m+1-i,j)$ and $C(i,j) = (i,n+1-j)$. Let $A_0$ be the subgroup generated by R,C and, only in a square, $T(i,j) = (j,i)$. These are explicit automorphisms. Define $H\pi = \{g \in A_0 : \chi(gv) = \chi(v) \text{ for every } v\}$. It is a subgroup, and every $H\pi$-orbit lies in a $\pi$-cell. We prove the stronger assertion MainOrbit: any related u,v are joined by an element of $H\pi$.

Lemma 2 (saturation). Degree is constant on cells. If S is a union of cells, so is $\{v : N(v) \cap S \neq \varnothing\}$. Indeed degree is the sum of all neighbour-colour multiplicities. The count into S is the sum over its cells, so is constant on each cell; being positive is consequently cell-invariant.

Lemma 3 (row determinism). If $\chi$ and $\psi$ obey the same rule M and agree on row 1, then $\chi = \psi$. Suppose they agree through row $i < m$. At (i,j), every neighbour except (i+1,j) lies in rows at most i. Cancel the multiset of these known neighbour colours from $M(\chi(i,j)) = M(\psi(i,j))$. The remaining singleton gives $\chi(i+1,j) = \psi(i+1,j)$. Induction completes the proof. Reflections and transposition give the same statement for any outer row or column. This argument also applies to one-vertex rows and to equality colours.

Lemma 4 (positive-layer propagation). Suppose $\lambda : V \to \{1,2,\ldots\}$ is constant on cells and invariant under $A_0$. Suppose each w with $\lambda(w) \geq 2$ has a neighbour a with $\lambda(a) = \lambda(w)-1$ for which w is the unique neighbour of a of layer at least $\lambda(w)$. If related vertices of layer 1 are joined by $H\pi$, then MainOrbit holds.

Proof. Induct on $\lambda(w)$, which is positive. Given $w \equiv w'$ of layer $k \geq 2$, choose a as stated. Equitability at w,w′ gives a neighbour a′ of w′ with $a \equiv a'$. By induction some $g \in H\pi$ sends a to a′. Transport the unique-neighbour property of a through g. The vertex gw is the unique neighbour of a′ of layer at least k, since g preserves $\lambda$. But w′ is such a neighbour too, so $gw = w'$. The converse inclusion of orbits in cells was already noted. Positivity is essential for the induction base.

Orbit partitions of any automorphism subgroup are equitable: if h maps u to v, then $w \mapsto hw$ bijects $N(u) \cap B$ with $N(v) \cap B$ for each orbit B. Thus only MainOrbit remains for the first assertion of Theorem 1.

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## 3. The saturated border and corner timing

Assume m,n ≥ 3. Define ℓ(i,j) = min(i,j,m+1−i,n+1−j), and B = {ℓ = 1}. Corners have degree 2, other border vertices degree 3, and interior vertices degree 4. Thus B is a union of cells. Every edge changes ℓ by at most 1. The sublevel sets satisfy

$$\{\ell \le k+1\} = \{\ell \le k\} \cup \{v : N(v) \cap \{\ell \le k\} \neq \varnothing\}, \quad k \ge 1.$$

To check the nontrivial inclusion, if ℓ(v) = k+1 is attained by i, then (i−1,j) has layer k. The three other minimum coordinates give the other directions. Lemma 2 inductively makes every sublevel set saturated, hence makes ℓ constant on cells. Reflections and square transposition preserve ℓ.

The same coordinate observation proves the uniqueness hypothesis of Lemma 4. If ℓ(w) = k+1 is attained by i, choose a = (i−1,j). Its other neighbours, when present, are (i−2,j) and (i−1,j±1), all of layer at most k. The target w is its unique neighbour of layer at least k+1. The other attaining directions follow by reflection or exchange of coordinates. Ties between minimum coordinates cause no problem.

Saturation of B gives a restricted neighbour-colour rule $M_B(c) = ⟦\chi(w) : w \in N(v) \cap B⟧$ for v ∈ B of colour c. An internal vertex of any side has exactly two border neighbours, the preceding and following side vertices. Its third neighbour is interior, also when a dimension is 3. Corners have their two side neighbours in B. Thus the induced border graph is a cycle whose marked vertices are the four corners.

Lemma 5 (non-backtracking walks). Let $x_0,\dots,x_d$ and $y_0,\dots,y_d$ be side walks with non-corner internal vertices and $x_{t+1} \neq x_{t-1}$, $y_{t+1} \neq y_{t-1}$ at every internal step. If $\chi(x_0) = \chi(y_0)$ and $\chi(x_1) = \chi(y_1)$, their colours agree at all indices. At an internal step, the two distinct border neighbours are precisely the preceding and following vertices. Hence

$$M_B(\chi(x_t)) = ⟦\chi(x_{t-1}), \chi(x_{t+1})⟧.$$

Equality of the rules and cancellation of the preceding colour forces equality of the following colour. Induction proves the lemma, even if all these colours coincide.

Corollary 6 (corner timing). Two monotone side walks with matching first two colours reach their next corners at the same step. Compare their common prefix up to the earlier corner arrival using Lemma 5. If arrival times differed, equal colours would join a degree-2 corner and a degree-3 non-corner, contrary to Lemma 2. For walks starting at a corner, “next” means the opposite endpoint of that side. For walks starting internally, it means the first endpoint in the chosen direction. This convention includes arrival at step 1 on sides of length 3.

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## 4. Border transitivity by explicit symmetries

We describe every symmetry used in the proof. From the upper left corner (1,1), $R^aC^b$, with $a,b \in \{0,1\}$, maps the flag ((1,1),(1,2)) to the horizontal flag at the chosen corner, directed into its row. In a square, $R^aC^bT$ maps it to the vertical flag directed into its column. These eight maps realize all directed corner flags in a square. In a nonsquare only a horizontal side has n vertices, so a flag with that side length is realized by one of the first four maps.

Proposition 7. Related border vertices $u \equiv u'$ are joined by some $g \in H\pi$.

First suppose u = (1,1). Its two neighbours are p = (1,2) and q = (2,1). At u′ the equal neighbour multiset permits a choice of its two neighbours p′,q′ with matching colours, including $\chi(p') = \chi(p)$. The walk along row 1 from u through p has its next corner at step n−1. The side walk from u′ through p′ has next corner at step s−1, where s is the number of vertices of that side. Corner timing gives s = n. The preceding explicit flag maps therefore supply $g \in A_0$ taking u to u′ and p to p′. The two walks match along the whole first row by Lemma 5. Thus $\chi \circ g$ and $\chi$ agree there, and Lemma 3 implies $\chi \circ g = \chi$ everywhere. Hence $g \in H\pi$.

Next let u = (1,j) with $2 \le j \le n-1$. Equal restricted border rules at u,u′ match its forward and backward neighbours to distinct border neighbours a′,b′ of u′, respectively. These are vertices, so their distinctness does not require distinct colours. The walks from u in the two directions first reach corners at distances n−j and j−1. Apply corner timing to the matched walks from u′. Their endpoint distances are the same. Consequently u′ lies on a side of n vertices, at distance j−1 from the corner c′ reached through b′. Choose the explicit flag map g taking (1,1) to c′ and (1,2) to the next vertex into that side. Then g(1,j) = u′ and g(1,j+1) = a′. Lemma 5 in both directions gives $\chi \circ g = \chi$ on row 1, and row determinism gives it everywhere.

For completeness, the coordinate normalizations above are justified as follows. If $f : G \to G'$ is a graph isomorphism, transport colours by $\chi'(z) = \chi(f^{-1}z)$. A witness g′ for χ′ returns to $g = f^{-1}g'f$. Direct substitution gives gu = u′ and $\chi(gv) = \chi(v)$. Row and column reflections normalize $A_0$. Transposition between rectangular grids conjugates R to C and C to R; in a square it also conjugates T to T. Thus these transports preserve the required subgroup membership, rather than assuming it. They move any chosen side to the first row and any corner to (1,1).

Lemma 4 with $\lambda = \ell$ now proves MainOrbit for every $m,n \ge 3$, including 3×3. Nothing in the border argument excludes a side of length 3.

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## 5. Paths, ladders and exceptional square witnesses

Paths. For $m = 1, n = 1$ there is one vertex and the assertion is immediate. For $m = 1, n \geq 2$ put $\lambda(j) = \min(j, n+1-j)$. Layer 1 is exactly the degree-1 endpoints. The sublevel recurrence of Section 3 and Lemma 2 make $\lambda$ cell-invariant. Reversal preserves it. A target on the nearer left side has predecessor $j-1$, whose only possible other neighbour $j-2$ has smaller layer; reflect for the other side. If the endpoints are related, reversal preserves the endpoint colour. Neighbour-rule cancellation along the path then preserves every colour: at each step cancel the preceding colour to obtain the next. If the endpoints are unrelated, each endpoint cell is already a singleton. In either case layer-1 transitivity holds, and Lemma 4 proves MainOrbit. Transposition handles all $n = 1$ grids.

Ladders. Suppose $m = 2, n \geq 3$ and $\lambda(i,j) = \min(j, n+1-j)$. Layer 1 consists of the four degree-2 corners; other vertices have degree 3. Again sublevel saturation gives cell invariance. If $\lambda(i,j) = j \geq 2$, choose $a = (i, j-1)$. The other neighbours $(i, j-2)$ and $(3-i, j-1)$, if present, have layer at most $j-1$. Reflection gives the corresponding right-side choice. Hence Lemma 4 applies after checking corners.

Given related corners $u, u'$, let $u^*$ denote the other vertex in $u$'s end column. Of $u$'s two neighbours, $u^*$ has degree 2 and the neighbour towards the next column has degree 3. Degree separation and equitability force $u^* \equiv u'^*$. Select R exactly when the row coordinates of $u, u'$ differ, and C exactly when their end columns differ. Their product $g \in A_0$ sends $u$ to $u'$ and $u^*$ to $u'^*$. Therefore $\chi \circ g$ agrees with $\chi$ on the entire end column of $u$. Column determinism (Lemma 3 after transposition and, if needed, an end reflection) makes $g$ colour-preserving globally. This proves the base case and then MainOrbit for all ladders. Transposition gives $n = 2, m \geq 3$.

The 2×2 square. The graph is the four-cycle. Choose related $u, u'$ and one neighbour $p$ of $u$. Equitability gives a neighbour $p'$ of $u'$ of matching colour. A map among $R^aC^b$ and $R^aC^bT$ takes the directed edge $(u,p)$ to $(u',p')$. Follow the cycle in that direction. At each step cancel the known previous neighbour colour from the two-neighbour rule. All four colours agree under the map, which therefore lies in $H\pi$. This supplies MainOrbit without a computation.

The 3×3 square is already covered by Sections 3–4. In the supplied formal architecture it is also handled by a finite witness certificate: classify the four corners and four edge midpoints, keep the centre separate by degree, and check the eight displayed symmetries on every equitable candidate. These are alternative proofs for that branch. The ordinary argument here and the formal finite certificate both concern every equitable partition, rather than selected examples.

Combining these cases proves assertion (i) of Theorem 1 for all positive dimensions. The path proof is included for a self-contained endpoint, without a novelty claim for paths.

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## 6. Free vertices in the full automorphism group

The distinction between $A_0$ and the full group matters for stabilizers. Here is a deduction from MainOrbit that avoids assuming the classical automorphism-group classification.

Lemma 8. If x has trivial $A_0$-stabilizer, it has trivial Aut(G)-stabilizer, and Aut(G) = $A_0$.

Let S = Stab_{Aut(G)}(x). Its orbit partition is equitable, hence by MainOrbit is the orbit partition of some K ≤ $A_0$. The S-orbit of x is {x}, so every element of K fixes x. Triviality of the $A_0$-stabilizer gives K = 1. Every S-orbit is therefore a singleton. The natural action of graph automorphisms is faithful, so S = 1. Apply MainOrbit again to the orbit partition of Aut(G). For $g \in \mathrm{Aut}(G)$, there is $k \in A_0$ with $gx = kx$. Thus $k^{-1}g$ fixes x and is the identity, proving $g = k \in A_0$.

For $m \neq n$ with $m,n \geq 2$, $A_0$ consists of 1,R,C,RC: R and C commute and are involutions. Their four images of (1,1) are distinct corners, so this vertex has trivial $A_0$-stabilizer. When exactly one dimension is 1, the reflection in that coordinate is the identity and the other reflection interchanges the two endpoints. An endpoint has trivial stabilizer. When m = n = 1, the full group is trivial and the unique vertex is free.

For $m = n = k \geq 4$, the relations $T^2 = R^2 = C^2 = 1$, RC = CR, and TRT = C put every element of $A_0$ into the form $R^aC^b$ or $R^aC^bT$. The images of x = (1,2) under these eight maps are

(1,2), (k,2), (1,k−1), (k,k−1), (2,1), (k−1,1), (2,k), (k−1,k).

They are pairwise distinct for $k \geq 4$. Hence x has trivial $A_0$-stabilizer. Lemma 8 gives a free vertex in the full group in every claimed positive case.

At k = 2 every diagonal vertex is fixed by T and each off-diagonal vertex by the anti-diagonal reflection $(i,j) \mapsto (3-j,3-i)$. Both are nonidentity automorphisms. At k = 3, the corners (1,1),(3,3) and the centre are fixed by T; the other corners (1,3),(3,1) are fixed by $(i,j) \mapsto (4-j,4-i)$. The two middle-row edge vertices are fixed by R, and the two middle-column edge vertices by C. Thus every vertex in either exceptional square has a nontrivial full-group stabilizer. This proves assertion (iii).

The supplied formal proof proves the positive classification independently of MainOrbit. Its oriented-flag rigidity follows a side because the inward neighbour has degree 4, the next side neighbour has degree at most 3, and injectivity excludes the preceding vertex. Unequal side lengths contradict corner timing. A fixed nonsquare corner cannot interchange directions; a fixed near-corner in a square of side at least 4 fixes its unique adjacent corner. Row determinism then fixes the whole grid. The path and ladder cases use their endpoint and degree-separated end-column versions. These differences in dependency order do not change the theorem being proved.

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## 7. Lattices and the two cardinality obstructions

The equitable partitions form a complete lattice in refinement order. Here is an argument valid for any finite graph. Given equitable setoids r in a family S, let s be their join in the lattice of all setoids. For each $r \le s$, projecting the r-neighbour-colour multiset onto s-classes makes the s-neighbour multiset equal at r-related vertices. The kernel t of the map sending a vertex to its s-neighbour multiset therefore contains every r. The universal property of the join gives $s \le t$, which is exactly equitability of s. This includes the empty join, the discrete relation. Meets within EqPart are joins of equitable common refinements. One must not identify them with ordinary setoid intersections without proof. The subgroup lattice has intersections as meets and generated unions as joins.

By assertion (i), $\Phi : \mathrm{Sub}(\mathrm{Aut}(G)) \to \mathrm{EqPart}(G)$ is surjective. It is monotone since $H \le K$ implies $Hv \subseteq Kv$. If x is free and $\Phi(H) \le \Phi(K)$, then for every $h \in H$ some $k \in K$ satisfies $hx = kx$. The element $k^{-1}h$ fixes x, hence $h = k \in K$. Thus $\Phi$ reflects order and is an order isomorphism. Order isomorphisms preserve all existing suprema and infima. This proves the free direction of assertion (ii), including its natural-map strengthening.

For the converse we give finite cardinality bounds. At each $k = 2,3$ the eight displayed symmetries form a dihedral subgroup D of the full group. Write $r(i,j) = (j,k+1-i)$ and $s = T$; then $r^4 = s^2 = 1$ and $srs = r^{-1}$. Ten distinct subgroups are $\{1\}$, $\langle r^2\rangle$, the four $\langle r^a s\rangle$ $(0 \le a < 4)$, $\langle r\rangle$, $\langle r^2, s\rangle$, $\langle r^2, rs\rangle$, and D. Distinctness is checked by their elements in these normal forms. Thus the full subgroup lattice has at least ten elements.

An economical upper bound on partitions uses the proved MainOrbit. D is exactly $A_0$ here and has exactly those ten subgroups: a subgroup's intersection with $\langle r\rangle$ is 1, $\langle r^2\rangle$ or $\langle r\rangle$; if it contains a reflection, all its reflections constitute one coset of that intersection. At 2×2, $\langle r\rangle$, $\langle R,C\rangle$ and D have the same one-cell orbit partition, saving two images; moreover $\langle r^2\rangle$ and $\langle r^2,T\rangle$ have the same opposite-corner-pair partition, saving another. Hence $|\mathrm{EqPart}| \le 7$. At 3×3, $\langle r\rangle$ and D share the centre/corners/edge-midpoints partition, so $|\mathrm{EqPart}| \le 9$. Since $7,9 < 10$, neither exceptional lattice is even equinumerous with the subgroup lattice. This proves (ii).

| Grid | Equitable partitions | Full-group subgroups |
|---|---|---|
| 1×1 | 1 | 1 |
| 1×2, 1×3 | 2 | 2 |
| 2×2 | 7 | 10 |
| 2×3, 3×4 | 5 | 5 |
| 3×3 | 9 | 10 |
| 4×4 | 10 | 10 |

These agree with the prior independent enumeration [4]. The exceptional upper bounds in Lean instead follow from exhaustive four-point setoid classification: 15 candidates at 2×2 and $15^2$ degree-separated candidates at 3×3, of which 7 and 9 satisfy equitability. It constructs ten actual full-group subgroups separately, without MainOrbit or a full-group cardinality assumption.

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## 8. Current endpoint and historical proof rounds

The source-only packet contains 47 modules in src/, including six auditors; four additional Lean files are negative controls, not part of that module count. Coordinates are zero-based Fin m × Fin n. G is literally Mathlib's (pathGraph m).boxProd (pathGraph n). IsEquitable uses Nat.card of the subtype {x : G.Adj u x ∧ r x w}, for every representative w of every cell. Finiteness identifies this count with Definition 5.1. neighborCells uses Quotient r and a multiset, and equitable_iff_neighborCells proves the equivalence retaining multiplicities.

EqPart is the subtype of equitable setoids, with relation inclusion as its order. orbitSetoid quantifies over actual subgroup elements of G ≃g G. HasFreeVertex uses that full group. Hpi consists of $A_0$ elements preserving every cell individually. The complete lattice construction in GridLattice is independent of MainOrbit. The final endpoint is GridFinal6.mathdb_343536, quantified over all natural m,n with 1 ≤ m and 1 ≤ n. It has no residual geometric or combinatorial premise.

| Historic-al round | Added proof content | Boundary then (not current) |
|---|---|---|
| 1 | GridBasic, GridSymmetry, GridChecks: definitions, orbit equitability, free-vertex order reflection, explicit symmetries, regressions. | GridFinal: MainOrbit, FreeClassification, ExceptionalNoIso remain premises. |
| 2 | GridNeighbours, GridPropagation, GridDeterminism, GridTransport, GridLattice: multisets, positive layers, row determination, transport, complete lattice. | GridFinal2: the same three premises. |
| 3 | GridFiniteCodes and exceptional modules: complete candidate classification, upper bounds 7/9, ten subgroup witnesses; GridFreePaths. | GridFinal3: MainOrbit and FreeClassification remain. |
| 4 | GridDegrees, GridSideRigidity, GridFreeClassification: full-group free classification; GridOrbitPaths, GridTransposeGroup, GridOrbitSmall: paths, transpose membership, exceptional orbits. | GridFinal4: only MainOrbit remains. |
| 5 | GridLadderLayers, GridLadderOrbit: all ladders; GridOrdinaryLayers, GridBorderWalk, GridSideColors: layers, saturated border, non-backtracking walks and timing. | GridFinal5: only OrdinaryBorderTransitivity remains. |
| 6 | GridColorFlags, GridColorTiming, GridCornerColors, GridSideAlignment, GridInternalFlags, GridBorderFrames, GridOrdinaryBorder: explicit ordinary border witnesses. GridFinal6 assembles the theorem. | No remaining premise. |

In particular ordinary_border_transitivity discharges the round-5 obligation, and main_orbit_proved has the exact frozen type MainOrbit. GridFinal6 invokes GridFinal4 with this theorem; GridFinal4 supplies the already proved free_classification and exceptional_no_iso. The stronger natural statement follows by substituting these same proofs into natural_iso_iff and natural_iso_of_free from GridFinal. Nothing is inferred merely from the endpoint's name.

Historical comments and copied round documents sometimes call obligations “unsolved” or the implementation “partial.” Those describe earlier boundaries only. They are preserved byte-for-byte in the frozen evidence. The table above and this draft's unconditional endpoint description supersede those historical headlines; the frozen inputs themselves have not been edited.

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## 9. Audits, reproduction and trust limits

The supplied round-6 validation [5] reports strict serial compilation with lean -j1 -M3500 -DwarningAsError=true and explicit-module leanchecker checks for all 47 src modules with allowOpaque enabled. Lean is pinned to 4.34.1 and mathlib to d13f23b723b8a846827a245b89c10fc7d3f11612. The archive SHA-256 is recorded in the dossier as

a4068de99037028f817f2f5d1652a97d7508c4419353da95abea04b45b977953.

Audit6 reports 299,549 inventoried and audited declarations (equal sets), including 742 LOCAL declarations, with no LOCAL unsafe, partial, axiom, unsafe_rec, implemented_by or extern declarations. Inventoried and audited declarations agree, including generated declarations and opaque values; the endpoint import closure is inventoried separately. The reported transitive axioms of the endpoint, main_orbit_proved and ordinary_border_transitivity are exactly propext, Classical.choice and Quot.sound. The imported compiler and tactic infrastructure has upstream unsafe declarations and additional axioms in its inventory. This is not a claim that the whole imported environment is axiom-free; the theorem dependency audit is the relevant narrower assertion.

The fidelity review adds that auditor imports emit deprecation warnings despite the requested warningAsError flag and successful exit status. Here “strict” denotes the actual flags and successful checks, not a warning-free transcript.

The four inherited negative controls are intended to fail: FalseIsoThree asserts the false exceptional lattice isomorphism at 3×3; bad-partition uses a non-equitable partition; free-three asserts a free vertex in 3×3; reverse-order tests the wrong partition-order direction. Finite exceptional computations use kernel decide rather than native_decide. The final fidelity review independently enumerated all degree-compatible partitions and full adjacency-preserving permutations, then all subgroups, for 1×n ($1 \le n \le 6$), 2×2, 2×3, 3×3 and 3×4. It compared the actual partition sets with subgroup orbit sets and compiled separate kernel-decided sanity witnesses. These finite checks are diagnostic, not a proof of the quantified statement.

The replay driver locally recompiles the missing pinned mathlib Bound.Attribute artifact into its local build tree. Upstream package directories are inputs, not write destinations. This artifact is part of the trusted compilation setup and must be disclosed rather than described as a pristine prebuilt mathlib replay. The exact scripts and tool paths are retained in the source packet.

There was a documented round-6 resource deviation: a development audit briefly overlapped the fresh replay, was terminated, and was excluded from certification. Final validation is reported as serial. Perfect session-wide compliance for that historical lane is not claimed. The resource qualification remains for root to assess; it is not a mathematical hypothesis.

This revision lane did not run Lean, lake or leanchecker. It reused the source ZIP byte-identically and tested safe extraction, rejecting unsafe names, links, duplicates and compiled artifacts. Serial document checks use the inherited capped runner: two CPUs, sampled process-group RSS 6 GiB, CPU 120 seconds, wall 140 seconds, MALLOC_ARENA_MAX=2, with lane-local temporary/cache paths. Orientation reads preceded input binding; short setup/authoring calls were affinity/CPU limited but not all RSS-sampled. No perfect all-call enforcement is claimed.

The dossier includes raw MathDB and cached source evidence, frozen implementation/review records, editable paper, PDF, extraction and page-QA receipts, and manifests. REPRODUCE.md separates document reproduction from preserved replay instructions for root. Packaging and reported validation confer no accepted credit.

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## 10. Related work, access limits and draft status

The mathematical source is Neuberger, Sieben and Swift [1], arXiv:1908.05797 v1 (15 August 2019), v2 (18 February 2020), later published in SIAM Journal on Applied Dynamical Systems. Its free-vertex remark already supplies the key injectivity observation for the lattice consequence. This proof addresses finite grids with free boundary; no claim about infinite lattices or periodic networks is intended.

The earlier independent review [4] reports Brito and Manoel's tree-network theorem [2] from its abstract: balanced colourings of trees are symmetry-induced. In particular the path case is already covered. We do not claim priority for paths. The new mathematical content under consideration is assertion (i) for $m,n \geq 2$. The lattice conclusion is a consequence and the automorphism-group description is classical. The result is a short elementary resolution of a published conjecture, of modest significance; a formal certificate does not by itself establish novelty or significance.

The review also records the compact-graph channel via Godsil [3]: a result establishing grid compactness would imply the first assertion. That channel was not exhaustively excluded. The earlier review's bounded search of citing works and the authors' follow-ups found no earlier proof for finite grids; the fidelity review performed no new literature search. Bodies concerning infinite or periodic lattices do not automatically transfer to free boundaries. No priority claim is made here.

This lane used cached source material and local review records only. The fidelity review read cached arXiv v1/v2 text and local records, performed no new literature search and did not refresh public registries. It performed no network or browser access. Neither this lane nor the fidelity review read the SIADS, Brito–Manoel or Godsil bodies. The earlier review likewise records body-access limits. Bibliographic DOIs below are verified only to the extent documented in its saved metadata receipts, not by a new publisher fetch. MathDB's 2018 reference to arXiv:1807.10954 is unrelated; the relevant source conjecture is the 2019 preprint, journal 2020.

The root registry refresh of 3 November 2026 records OpenAI head 4eaee87548250b2682affda7d05d947c05554bae and Omega head bc417b2da78b2e300375b9df9ed6a22f1fe36c29. Its complete-tree path screen found zero relevant hits in Omega D5 and Problems; this is not a full-body proof search. These are received root records, not this lane's independent live search.

Final received records: grid-fidelity-r1/REVIEW.md and review.json give A–C PASS and D–E QUALIFIED_PASS. Its interrupted Audit6 run was excluded and rerun; auditor import warnings and historical resource qualifications remain disclosed. The separate decisive root receipt grid-parent-final-r1/replay-result.json reports PASS: 47/47 modules strictly compiled and explicitly leanchecked, 4/4 controls rejected, Audit6 inventory = audited (299,549; LOCAL 742), and standard axioms only. Root's comment/string-aware scan found no forbidden tokens in all 47 source modules. Root corrected its counter to include the first PASS line. Its registry-refresh.json reports CLEAN at Omega bc417b2d, within the path-screen scope above. Campaign status is 29/40 beyond 39; this revision adds zero accepted credit. Root reads the paper and decides.

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## Bibliography

[1] J. M. Neuberger, N. Sieben and J. W. Swift, Invariant synchrony subspaces of sets of matrices, SIAM J. Appl. Dyn. Syst. 19(2) (2020), 964–993. arXiv:1908.05797, v1/v2. DOI: 10.1137/19M1283495 (metadata receipt in [4]).

[2] N. Brito and M. Manoel, A Symmetry-Based Classification of Synchrony in Tree Networks, J. Nonlinear Sci. (2026). DOI: 10.1007/s00332-026-10297-1 (abstract/metadata evidence in [4]; body not accessed).

[3] C. D. Godsil, Compact graphs and equitable partitions, Linear Algebra Appl. 255 (1997), 259–266. Bibliographic record in [4]; body not accessed. No DOI asserted.

[4] Independent adversarial review, grid-review-r1, REVIEW.md and review.json, 3 November 2026. Local research record, not a publication.

[5] Grid formalization rounds 1–6, source-only round-6 packet and validation records; independent grid-fidelity-r1 review and grid-parent-final-r1 root records as snapshotted in this dossier. Local research artifacts, not publications.

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