Truncated exponentials defeat every subunit Rényi threshold

Research manuscript

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Abstract. For each real order $0 < p < 1$, we prove that a finite normalized truncation of the rate-one exponential has a strictly larger iid positive-sum Rényi entropy increment than the rate-one exponential itself. The proof retains the input power integral and uses an exact translated deficit identity followed by dominated convergence. Consequently no threshold strictly below one can have the exponential-maximizing property proposed in the unnumbered paragraph after (1.2) on printed page 2 of Fu–Li [1]. We make no assertion about their Shannon Conjecture 1.1, the cube branch, or actual optimizers. The accompanying source-only certificate proves the literal density comparison for every real subunit order.

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1. Statement and source scope

For a probability density f on the real line and $p \in (0,1)$, write $M_p(f) = \int_{\mathbb{R}} f(x)^p\,dx$ and $h_p(f) = \log M_p(f)/(1-p)$. All logarithms are natural. We use these expressions only when the input and output power integrals are finite and strictly positive. Convolution means the literal Lebesgue integral $(f*g)(s) = \int_{\mathbb{R}} f(t)g(s-t)\,dt$. Thus for independent random variables X,Y with common density f, their positive sum has density $f*f$, and its increment is $\Delta_p(f) = h_p(f*f) - h_p(f)$.

Fu and Li [1, printed p. 2, unnumbered paragraph after (1.2)] suspect a constant $p_0 \in (0,1)$ with exponential maximization above the threshold and cube-uniform maximization below it for iid log-concave vectors. Dimension one is already part of that proposal. Its necessary consequence that the rate-one exponential maximizes Δ_p for $p_0 < p < 1$ is enough for the contradiction below. This source assertion is distinct from their numbered Shannon Conjecture 1.1.

Theorem. For every real p with $0 < p < 1$, there exists a finite $a > 0$ such that the log-concave probability density $f_a(x) = c_a e^{-x} 1_{[0,a]}(x)$, where $c_a = (1-e^{-a})^{-1}$, satisfies $\Delta_p(f_a) > \Delta_p(e)$, where $e(x) = e^{-x}1_{[0,\infty)}(x)$. All four entropies in this comparison are finite.

The theorem concerns each fixed real order, with a cutoff depending on that order. No uniform cutoff as p tends to one is required or asserted. We compare the displayed rate-one density only; no all-rate theorem is part of this paper or its certificate. We do not classify maximizers or establish any cube-uniform assertion.

2. Literal normalization and convolution

Fix $a > 0$. Since $\int_0^a e^{-x}dx = 1-e^{-a}$, the displayed c_a normalizes f_a to mass one. Its potential is $x - \log c_a$ on $[0,a]$ and $+\infty$ outside that interval, a convex extended-valued function. Thus f_a is globally log-concave, including its zero region. Endpoint conventions affect no integral. Taking the product density $f_a(x)f_a(y)$ supplies genuinely independent, identically distributed coordinates.

At a sum value s, the overlap interval is $[0,a] \cap [s-a,s]$. The integrand on that interval is the constant $c_a^2 e^{-s}$. Its length is the nonnegative tent $T_a(s)$, equal to s for $0 \le s \le a$, to $2a-s$ for $a \le s \le 2a$, and to zero otherwise. Hence, pointwise on the whole line,

$$(f_a * f_a)(s) = c_a^2\, e^{-s}\, T_a(s). \quad (1)$$

This output is a probability density by Tonelli's theorem applied to the nonnegative product density. Both f_a and its convolution are bounded and compactly supported, and their p-power integrals are finite and positive for $p > 0$. For

the baseline, direct convolution gives $(e*e)(s) = se^{-s}$ on $s \ge 0$ and zero elsewhere. Its positive-order power integral is finite by exponential decay.

Define the strictly positive finite integral J and the truncated tent integral J_a by

$$J = \int_0^\infty e^{-ps} s^p \, ds, \; J_a = \int_0^\infty e^{-ps} T_a(s)^p \, ds. \quad (2)$$

In particular J_a is defined by the actual tent, independently of any desired deficit estimate. Integrating the input power and using (1) gives

$$M_p(f_a) = c_a^{\,p}(1-e^{-pa})/p, \; M_p(f_a * f_a) = c_a^{\,2p} J_a. \quad (3)$$

$$R_a := M_p(f_a * f_a)/M_p(f_a) = c_a^{\,p} p J_a/(1-e^{-pa}). \quad (4)$$

For e the input moment is 1/p and output moment is J, so the baseline ratio is R = pJ. The denominator $1-e^{-pa}$ and the remaining factor $c_a^{\,p}$ in (4) are essential. Comparing output entropies alone would not compare increments.

3. Exact deficit and the subunit-order limit

Set $D_a = J - J_a$. The integrands agree on [0,a]. Subtracting their integrable tails and substituting s = a+u yields the exact identity

$$D_a = e^{-pa} \int_0^\infty e^{-pu} [(a+u)^p - (a-u)_+^{\,p}] \, du, \quad (5)$$

where $t_+ = \max(t,0)$. Indeed the shifted tent $T_a(a+u)$ is $(a-u)_+$, including the region u > a where it vanishes. There is no subtraction of divergent integrals in (5).

For every $a,u \ge 0$ and $0 < p < 1$, the bracket $B_a(u)$ in (5) satisfies

$$0 \le B_a(u) := (a+u)^p - (a-u)_+^{\,p} \le (2u)^p. \quad (6)$$

Nonnegativity follows from monotonicity of the positive power. If $u \le a$, use subadditivity $(v+w)^p \le v^p + w^p$ for $v,w \ge 0$, with v = a−u and w = 2u. If u > a, then $(a-u)_+ = 0$ and $a+u \le 2u$. This proves the same upper bound in the remaining region.

For fixed $u \ge 0$ and a > u, the mean value theorem, or the tangent inequality for t^p, gives

$$0 \le B_a(u) \le 2pu(a-u)^{p-1} \to 0 \text{ as } a \to \infty. \quad (7)$$

At u = 0 this is immediate. At u > 0 the exponent p−1 is negative, so the right side tends to zero. The bound $e^{-pu}(2u)^p$ from (6) is integrable on $(0,\infty)$: near zero it is a constant times u^p, and at infinity exponential decay dominates that power. The dominated convergence theorem therefore proves, for this fixed p,

$$K_a := \int_0^\infty e^{-pu} B_a(u) \, du \to 0, \; D_a = e^{-pa} K_a. \quad (8)$$

Since J > 0, all sufficiently large finite a > 0 satisfy $K_a < J$. Equation (8) then implies

$$J_a > (1-e^{-pa})J. \quad (9)$$

As $c_a > 1$, equations (4) and (9), with their positive denominator, yield $R_a > pJ = R$. The moments in these ratios are positive. Consequently the logarithmic identity

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$$\Delta_p(f) = \log(M_p(f*f)/M_p(f))/(1-p) \quad (10)$$

and $1-p > 0$ give $\Delta_p(f_a) > \Delta_p(e)$, completing the theorem. This argument uses standard power concavity, exponential integrability, translation invariance of Lebesgue integration, and dominated convergence. These classical tools, including Tonelli and the mean value theorem, are not novelty claims.

4. Threshold consequence and comparison with prior results

Suppose the necessary threshold consequence held for some $0 < p_0 < 1$: every admissible log-concave real density f would satisfy $\Delta_p(f) \le \Delta_p(e)$ whenever $p_0 < p < 1$. Take $p = (1+p_0)/2$. It lies strictly between p_0 and one. The theorem supplies an admissible f_a violating that inequality. This refutes the necessary consequence, and hence the proposed full threshold. Nothing in this reasoning treats $p = 1$, the Shannon conjecture, or the identity of actual optimizers.

Theorem 3.4 and Lemma 3.7 of [1] compare output entropies under a matched input min-entropy. They do not compare output-minus-input Rényi entropies. Their matched exponential input need not have the same order-p entropy as f_a. Thus subtracting the changing input entropy from the output bound cannot establish the proposed increment inequality. Formula (4) exhibits the missing normalization explicitly.

Melbourne and Tkocz [2, Theorem 6.1 and Corollary 6.2 in the retained preprint] treat entropy comparisons and difference inequalities, using the identity between the maximum density of $X-Y$ and the input collision moment. That mechanism does not furnish a strict improvement for our positive-sum quotient at every $p \in (0,1)$. The truncated exponentials here are not symmetric, so reflection cannot identify their positive sums with differences. Melbourne, Tkocz and Wyczesany [3, Theorem 2 and Remark 15] study point evaluation and localization under a variance constraint; their use of truncated exponentials is prior mathematics, but it does not itself optimize the nonlinear quotient in (4). Their higher-order anti-concentration comparisons have different order ranges and objectives.

The project reviews found no complete published bridge or exact prior refutation in the bounded primary bodies and pinned registry material inspected. This is a bounded clearance, not a worldwide priority claim. The proof is elementary and the family familiar; a prior first-variation or truncation argument for this same quotient could defeat novelty. The retained registry review discloses selective baseline semantic coverage and 51 historical whole-source context-hash mismatches; exact content-addressed governing and primary atoms were recovered, but entire old source versions were not reconstructed. Independent final review, a later bounded freshness check and root acceptance remain pending; candidate credit is zero.

5. Formal scope and reproducible source packet

The certificate uses real Lebesgue integrals and literal definitions in namespace Renyi: density, exponential, convolution, moment and entropy. Admissible p f records nonnegativity, global multiplicative log-concavity, input integrability and mass one, integrability and positivity of the input p-power, integrability of each convolution integrand, and nonnegativity, p-power integrability and positive moment of the output. Both displayed densities satisfy these requirements unconditionally for $p > 0$.

The declaration every_order_counterexample proves the theorem with only $0 < p$ and $p < 1$ as hypotheses; not_source_threshold negates SourceThresholdConsequence using the midpoint order above. Ja is the tent integral, and deficit_identity proves (5) from it. The 11 mathematical modules are audited through import all, loading their actual bodies. The audit enumerates all 104 declarations, including generated helpers, and records actual types, module origins, transitive axioms and safety flags. Only propext, Classical.choice and Quot.sound are allowed. No assumed analytical bridge or prebuilt local compiled module is included in the source archive.

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The packet binds Lean 4.34.1, Mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612 and all eight inherited auxiliary package pins. Its verifier checks source manifests, immutable input hashes, the compiler, package revision metadata, upstream .olean files and existing private/server companions, then builds serially with the unchanged bounded Lean guard. Receipts preserve the earlier informational-ring replay failure and other qualifications rather than presenting every historical attempt as successful. The source rebuild uses ReportLab's invariant PDF mode and fixed ZIP timestamps. Rendering and actual page inspection supplement the mathematical and declaration audits; they do not establish novelty or acceptance.

References

[1] Zhen Fu and Jiange Li. A reverse entropy power inequality for i.i.d. log-concave random variables. arXiv:2510.09206v3, 21 July 2026. Primary target: printed p. 2, unnumbered paragraph after (1.2); comparisons: Theorem 3.4, Lemma 3.7. https://arxiv.org/html/2510.09206v3

[2] James Melbourne and Tomasz Tkocz. Reversals of Rényi Entropy Inequalities under Log-Concavity. arXiv:2005.10930v1, 21 May 2020. Retained preprint Theorem 6.1, Corollary 6.2. https://arxiv.org/abs/2005.10930

[3] James Melbourne, Tomasz Tkocz and Katarzyna Wyczesany. A Rényi entropy interpretation of anti-concentration and noncentral sections of convex bodies. arXiv:2406.04200v2, 25 January 2025. Theorem 2, Remarks 14–15, Section 4. https://arxiv.org/abs/2406.04200v2

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