Connected graphs with edge-irregulator number $3|E|/8$

Research manuscript

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Abstract

We construct, for every integer $t \geq 2$, a connected simple graph with $6t + 1$ vertices, $8t$ edges and edge-irregulator number exactly $3t$. The excess above one third of its edge count is therefore $t/3$. This refutes Conjecture 1 of Bensmail, Catherinot, Fioravantes, Marcille and Oijid, which asks for a uniform additive constant on connected graphs. A two-part local charging argument gives the lower bound; an explicit three-edge deletion in each block attains it. The proof is independent of computer enumeration. A source-only Lean certificate accompanies this draft.

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1 Definition, source question and result

All graphs below are finite, undirected and simple. A graph H is *locally irregular* if $d_H(u) \neq d_H(v)$ for every edge $uv \in E(H)$. For $S \subseteq E(G)$, let $G - S$ retain all vertices and remove the edges in S. An *edge-irregulator* is such an S for which $G - S$ is locally irregular, and

$$I_{\mathrm{e}}(G) = \min\{|S| : S \subseteq E(G),\ G - S \text{ is locally irregular}\}.$$

The minimum exists since deleting all edges is allowed. In particular, the endpoint degrees in this definition are *recomputed in* $G - S$. Neither connectivity nor positive minimum degree of $G - S$ is required.

Conjecture 1 of [1, p. 356] states that there exists an absolute real $c \geq 1$ such that every connected graph G satisfies

$$I_{\mathrm{e}}(G) \leq |E(G)|/3 + c. \tag{1}$$

This is the publisher's connected-graph claim, with no assumption on the original minimum degree. It is distinct from assertions restricted to regular graphs or other special classes.

Theorem 1. *For every integer $t \geq 2$ there is a connected simple graph G_t such that*

$$|V(G_t)| = 6t + 1, \qquad |E(G_t)| = 8t, \qquad I_{\mathrm{e}}(G_t) = 3t.$$

Consequently no real constant $c \geq 1$ satisfies (1) *for every connected finite simple graph.*

2 Construction and the lower bound

Take one vertex h. For each $i \in \{1, \dots, t\}$ take six private vertices $a_i, b_i, c_i, d_i, x_i, y_i$. All these vertices are distinct, including across blocks. The edges are precisely the following eight edges for each i:

$$a_ib_i,\ a_ic_i,\ a_id_i,\ b_ic_i,\ b_id_i,\ c_ix_i,\ x_iy_i,\ y_ih. \tag{2}$$

The first five edges form a diamond (K_4 minus c_id_i); the last three form a path from c_i to h. Every vertex has a path to h, so G_t is connected. There are no loops or duplicate edges. Different blocks share only h, and their edge sets are disjoint. This proves the two claimed cardinalities.

Fix an arbitrary edge-irregulator S, and put $H = G_t - S$. Fix a block and omit its subscript. We make two disjoint charges against S.

Lemma 1. *At least two of ab, ac, ad, bc, bd belong to S.*

Proof. The only edge leaving the diamond is cx. Let $\varepsilon = 1$ if cx is retained and $\varepsilon = 0$ otherwise. The degrees of a, b, d in H are exactly their retained diamond degrees, and the degree of c is its retained diamond degree plus ε.

If no diamond edge is deleted, $d_H(a) = d_H(b) = 3$ contradicts the retained edge ab. If exactly one is deleted, there are five possibilities. The swap $a \leftrightarrow b$ preserves the diamond and its attachment at c, so the following three cases include all five possibilities:

1. The deleted edge is ab. Then a, b, d each have degree 2. The retained edge ad has equal endpoint degrees.

2. The deleted edge is ad (or, by the swap, bd). Then $d_H(a) = 2$, $d_H(b) = 3$, and $d_H(c) = 2 + \varepsilon$. For $\varepsilon = 0$, retained ac has degree pair $(2, 2)$; for $\varepsilon = 1$, retained bc has degree pair $(3, 3)$.

3. The deleted edge is ac (or, by the swap, bc). Then $d_H(a) = d_H(d) = 2$, contradicting the retained edge ad.

Each case violates local irregularity, for both possible attachment states. These degrees are those of the whole graph H; no exterior deletion can alter them except through the already treated edge cx. □

Lemma 2. *At least one of cx, xy, yh belongs to S.*

Proof. Otherwise x and y each have degree 2 in H, contradicting the retained edge xy. The final degree of h is irrelevant. □

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3 Sharpness and the source quantifiers

The two sets charged in Lemmas 1 and 2 are disjoint. Moreover, all eight-edge block sets in (2) are pairwise disjoint and partition $E(G_t)$. Thus the charges count actual deleted edges without multiplicity. For every admissible S,

$$|S| = \sum_{i=1}^{t} |S \cap E_i| \;\geq\; \sum_{i=1}^{t} (2+1) = 3t,$$

where E_i is the eight-edge set of block i. This proves $I_{\mathrm{e}}(G_t) \geq 3t$. The sharing of h does not identify any charged edges. This argument works against arbitrary deletions, including ones that isolate vertices or disconnect the retained graph.

For the reverse inequality, delete precisely

$$S_t = \bigcup_{i=1}^{t} \{a_i b_i, a_i d_i, x_i y_i\}.$$

The sets in this union are disjoint, so $|S_t| = 3t$. The private degrees in $G_t - S_t$, in the order $(a_i, b_i, c_i, d_i, x_i, y_i)$, are

$$(1, 2, 3, 1, 1, 1), \qquad d_{G_t - S_t}(h) = t.$$

Every retained edge is one of the following five types; the table gives its two final degrees:

edge	$a_i c_i$	$b_i c_i$	$b_i d_i$	$c_i x_i$	$y_i h$
final degrees	$(1,3)$	$(2,3)$	$(2,1)$	$(3,1)$	$(1,t)$

Every pair has unequal entries for $t \geq 2$. Therefore S_t is an edge-irregulator and $I_{\mathrm{e}}(G_t) \leq 3t$, establishing equality. The restriction $t \geq 2$ is used in the last pair. No optimum for $t = 1$ is asserted here.

To check the full negation of the source claim, let an arbitrary real $c \geq 1$ be proposed. By the Archimedean property choose a natural number $t > 3c$. Then $t \geq 2$, and

$$I_{\mathrm{e}}(G_t) - \frac{|E(G_t)|}{3} = 3t - \frac{8t}{3} = \frac{t}{3} > c.$$

This contradicts (1). Thus the construction defeats every proposed absolute additive constant, rather than only a particular numerical value. Equivalently the excess is unbounded along this connected family. □

Why the boundary calculation matters. An obstruction inside an induced subgraph need not remain an obstruction after edges to its exterior change the endpoint degrees. The diamond argument explicitly accommodates the only exterior contribution ε, while the path argument avoids the hub degree completely. This independence is what permits summing the lower bound over arbitrarily many blocks joined through one hub. The finite cases above supply the ordinary proof; an enumeration of masks is not a premise.

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4 Formal certificate and status

The accompanying certificate uses Lean 4.34.1 and pinned Mathlib with eight other pinned packages. The thirteen inherited mathematical modules are preserved byte for byte from the completed lower/source and upper-witness components. The added `Result.lean` combines them into the closed endpoints `EdgeIrregulator.minimum_exact` and `EdgeIrregulator.main`. The exact equality has only the hypotheses $t : \mathbb{N}$ and $2 \leq t$. `Source` defines actual graph edge deletion, the retained-degree predicate and the attained minimum. `Gadget`, `Edges` and `Degrees` define the graph and its edge indexing and neighbors. `Boundary`, `Local`, `Lower` and `Charges` connect the local constraints to arbitrary valid deletion sets and to pairwise disjoint actual edge charges. The local finite certificate is kernel checked. The self-contained cases on page 2 explain its mathematics. `UpperWitness`, `UpperDegrees` and `Upper` verify S_t, its cardinality and every retained degree and edge type. `Transport` proves `minimum_iso` for finite graph isomorphisms, and transports G_t to a graph on a finite ordinal. The finite-model equality is the declaration `EdgeIrregulator.fin_minimum_exact`; its module is `Result. Refutation` proves `source_refutation`, the negation of the literal claim with one real $c \geq 1$, every natural n, every connected simple graph on $\operatorname{Fin} n$, and existence of a deletion set with the claimed real bound. The closed main theorem includes connectivity, vertex and edge counts, both exact minima and this source negation.

The exhaustive audit enumerates all 167 declarations from fourteen mathematical module origins, including private and generated names. It records full types, transitive axiom dependencies and safety flags, and crosschecks the module declaration tables. Only `propext`, `Classical.choice` and `Quot.sound` are permitted. Audit metaprograms are outside the mathematical module set. The packet verifier checks the binary, all nine package revisions and imported package hashes before a serial strict rebuild from source.

Attribution and limits. The author line follows the campaign's established "Research manuscript" attribution; no personal identity is inferred. OpenAI Codex was used for proof and manuscript preparation, implementation and verification. This is a candidate draft, not an accepted or published paper. Candidate credit remains 0; campaign counts remain 7/30 additional and 37/60 total. One family is proposed, with no separate credit for helpers or individual parameters. Independent final root review and its final current prior-art refresh remain pending.

The existing bounded source/prior review dated 29 October 2026 examined the publisher definition and Conjecture 1, current registry inventories and selected actual claims. It found no same-object counterpart or complete subsumption bridge in that scope. It did not read every registry body or certify worldwide priority; unindexed and unpublished work remain outside its coverage. Exact review records, hashes, historical failures and resource limitations are included with the certificate. No new novelty clearance is claimed by this manuscript preparation.

References

[1] J. Bensmail, N. Catherinot, F. Fioravantes, C. Marcille and N. Oijid, *Graph Irregularity via Edge Deletions*, WALCOM 2026, pp. 353–367. Conjecture 1, p. 356. `https://doi.org/10.1007/978-981-95-7127-7_24`. Preprint: arXiv:2511.14514v1 (18 November 2025).