# Strict convexity of Gamma-ratio level curves

Research manuscript

October 28, 2026

**Abstract**

For every $c > 1$, the equation $\Gamma(h)\Gamma(h+2x)/\Gamma(h+x)^2 = c$ defines a unique positive height $h = h_c(x)$ for each $x > 0$. We prove that $h_c$ is strictly convex on the entire positive half-line. At $c = 2$, this answers the question in Remark 16(a) of Garrappa–Gerhold–Popolizio–Simon. The proof applies classical perspective convexity to Euler's product for the Gamma function; strictness follows from two consecutive factors of that product.

mathdb:2611.00024v1 04 Nov 2026

## 1 The level curve

**Theorem 1.** *Let $c > 1$. For each $x > 0$ there is a unique $h_c(x) > 0$ satisfying*

$$\frac{\Gamma(h_c(x))\Gamma(h_c(x)+2x)}{\Gamma(h_c(x)+x)^2} = c. \tag{1}$$

*For distinct $x_0, x_1 > 0$ and $0 < \lambda < 1$,*

$$h_c\big(\lambda x_0 + (1-\lambda)x_1\big) < \lambda h_c(x_0) + (1-\lambda)h_c(x_1).$$

For $c = 2$, the function in (1) is the function $h$ of [1, Lemma 15]. That paper proves its strict convexity on $[1, \infty)$ and proposes strict convexity on $(0, \infty)$ in Remark 16(a). The remark is unchanged in the published version of record. Its boundary classification for the Mittag–Leffler function in Theorem 23 uses this same curve, including at parameters below one. Theorem 1 concerns the shape of this curve; it makes no claim to resolve that paper's separate complete-Bernstein-function conjectures.

For $0 < x < s$, put

$$F(x,s) = \log\Gamma(s-x) + \log\Gamma(s+x) - 2\log\Gamma(s), \qquad g(u) = -\log(1-u^{-2}) \quad (u > 1). \tag{2}$$

Euler's product for the Gamma function gives

$$F(x,s) = \sum_{m=0}^{\infty} g\left(\frac{s+m}{x}\right). \tag{3}$$

For clarity, the corresponding finite approximants for $\log\Gamma(z)$ are

$$z\log N + \log(N!) - \sum_{m=0}^{N} \log(z+m).$$

In the combination (2), both $\log N$ and $\log(N!)$ cancel. Each remaining summand is $2\log(s+m) - \log(s-x+m) - \log(s+x+m)$, exactly the summand in (3). Passing to the limit proves the identity.

All summands in (3) are positive. For $m \geq 1$,

$$0 \leq g\left(\frac{s+m}{x}\right) \leq \frac{x^2}{(s+m)^2 - x^2} \leq \frac{x^2}{m^2}. \tag{4}$$

The first upper bound follows from $-\log(1-t) \leq t/(1-t)$ for $0 \leq t < 1$. Thus the tail converges uniformly for fixed $x$ and $s > x$. For fixed $x > 0$, $F(x,s)$ is continuous and strictly decreasing in $s$. Its limit is $+\infty$ as $s \downarrow x$, by the $m = 0$ summand, and zero as $s \to \infty$, by (4) and dominated convergence for the series. Consequently there is a unique $s > x$ with $F(x,s) = \log c$. Setting $h_c(x) = s - x$ proves existence and uniqueness in Theorem 1.

## 2 Perspective convexity and strictness

Direct differentiation gives

$$g'(u) = -\frac{2}{u(u^2-1)}, \qquad g''(u) = \frac{2(3u^2-1)}{u^2(u^2-1)^2} > 0 \quad (u > 1).$$

Hence $g$ is strictly convex. In particular, $xg((s+m)/x)$ is a classical perspective of $g$ composed with an affine shift. The perspective-convexity principle is standard [2, Propositions 2.3 and 2.13]; no new convexity principle is needed here. The positive normalization by $x$ is relevant: $xF(x,s)$, rather than $F(x,s)$ itself, is a sum of these convex perspectives.

We give the strict inequality explicitly. Let $x_0 \neq x_1$ be positive, $y_i = h_c(x_i)$, $s_i = x_i + y_i$, and write

$$a = \lambda, \quad b = 1-\lambda, \qquad X = ax_0 + bx_1, \quad S = as_0 + bs_1, \qquad A = \frac{ax_0}{X}, \quad B = \frac{bx_1}{X}.$$

Here $A, B > 0$, $A + B = 1$, and $X < S$. For every $m \geq 0$,

$$\frac{S+m}{X} = A\frac{s_0+m}{x_0} + B\frac{s_1+m}{x_1}. \tag{5}$$

Convexity of $g$ therefore gives

$$g\left(\frac{S+m}{X}\right) \leq Ag\left(\frac{s_0+m}{x_0}\right) + Bg\left(\frac{s_1+m}{x_1}\right). \tag{6}$$

At least one of $m = 0, 1$ gives strict inequality. Indeed, equality for both would force equality of both pairs of arguments, and subtraction yields $1/x_0 = 1/x_1$, contrary to $x_0 \neq x_1$. The convergent sums in (3) and the nonnegative termwise gaps in (6) imply

$$F(X,S) < AF(x_0,s_0) + BF(x_1,s_1) = \log c.$$

On the other hand, $F(X, X + h_c(X)) = \log c$. Strict decrease of $F(X,s)$ in its second variable gives $X + h_c(X) < S$. Subtracting $X$ gives

$$h_c(X) < ah_c(x_0) + bh_c(x_1),$$

which completes the proof of Theorem 1.



mathdb:2611.00024v1 04 Nov 2026

## 3 Formal and research status

Independent source, mathematical, formal, manuscript and bounded novelty reviews are complete, including reassessment of the explicit classical-perspective reduction. The result is one family for all $c > 1$; the Euler product, perspective theorem, Jensen inequality and two-factor strictness calculation are supporting tools, not separate outcomes.

The complete Lean certificate proves the literal Gamma-ratio statement without residual analytic premises. In namespace `GammaLevel`, the endpoint is `source_convexity`. The level-2 specialization is `source_remark16a`; existence and uniqueness are separately certified by `existsUnique_sourceLevel`. A fresh isolated source replay passed for all nine modules and 52 public declarations under Lean 4.34.1, Mathlib revision `d13f23b723b`, and warnings as errors. All audited declarations use only the standard axioms `propext`, `Classical.choice`, and `Quot.sound`. A separate independent final review reproduced the extracted replay and inspected every declaration type and axiom report. Numerical checks are supplementary evidence only; the campaign acceptance record is separate.

The bounded literature search found no earlier resolution of this level-curve question; worldwide priority is not asserted. The proof applies the classical perspective method to the Gamma-ratio equation.

## References


[1] R. Garrappa, S. Gerhold, M. Popolizio and T. Simon, *On some inequalities for the two-parameter Mittag–Leffler function in the complex plane*, J. Math. Anal. Appl. **551** (2025), 129588. doi:10.1016/j.jmaa.2025.129588. See also arXiv:2410.11852v4, Lemma 15 and Remark 16(a).

[2] P. L. Combettes, *Perspective functions: properties, constructions, and examples*, arXiv:1610.01552v3 (2017), Propositions 2.3 and 2.13.


mathdb:2611.00024v1 04 Nov 2026