Parameter concavity of the generalized inverse hyperbolic sine

Research manuscript

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Abstract

For each $x > 0$, the function $p \mapsto \int_0^x (1+t^p)^{-1/p}\,dt$ is strictly concave on $(\frac{1}{2}\log 2, \infty)$. This proves the parameter-concavity statement of Conjecture 5.1 in the 2019 survey of Yin, Huang, Wang and Lin. The proof combines an explicit second derivative with the classical binary-entropy bound. A literal Lean certificate verifies the integral statement using only the three standard logical axioms.

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For $p > 0$ and $x \geq 0$, write

$$A(p,x) = \operatorname{arcsinh}_p(x) = \int_0^x (1+t^p)^{-1/p}\,dt. \tag{1}$$

Conjecture 5.1 of [1] asks whether $p \mapsto A(p,x)$ is strictly concave on $(1,\infty)$ for every fixed $x \in (0,1)$. We prove a larger parameter range and allow every $x > 0$.

Theorem 1. *For every $x > 0$, the function $p \mapsto A(p,x)$ is strictly concave on $\left(\frac{1}{2}\log 2, \infty\right)$.*

Proof. For $u > 0$, define

$$H(u) = \log(1+u) - \frac{u\log u}{1+u}.$$

With $r = u/(1+u) \in (0,1)$ this becomes the binary entropy $-r\log r - (1-r)\log(1-r)$, so $H(u) > 0$. Also

$$H'(u) = -\frac{\log u}{(1+u)^2}.$$

Thus H increases up to $u = 1$ and decreases afterwards. In particular,

$$0 < H(u) \leq H(1) = \log 2. \tag{2}$$

Fix $t > 0$, and set $f(p,t) = (1+t^p)^{-1/p}$ and $u = t^p$. Direct differentiation gives

$$\partial_p \log f(p,t) = \frac{H(u)}{p^2}, \qquad \partial_p H(t^p) = -\frac{p(\log t)^2 u}{(1+u)^2}.$$

It follows that

$$\partial_p^2 f(p,t) = \frac{f(p,t)}{p^4}\left[H(u)^2 - 2pH(u) - \frac{p^3(\log t)^2 u}{(1+u)^2}\right]. \tag{3}$$

If $p > \frac{1}{2}\log 2$, then (2) yields $H(u) < 2p$. The first two terms in the bracket equal $H(u)(H(u)-2p) < 0$, and its last term is nonpositive. Hence $\partial_p^2 f(p,t) < 0$ for every $t > 0$ in the stated parameter range.

Take distinct $p_0, p_1 > \frac{1}{2}\log 2$ and $0 < \lambda < 1$. Strict concavity of $f(\cdot,t)$ gives, for every $t > 0$,

$$f\big(\lambda p_0 + (1-\lambda)p_1, t\big) > \lambda f(p_0,t) + (1-\lambda) f(p_1,t).$$

Each integrand is continuous on $[0,x]$, including $t = 0$, where its value is one. The Jensen gap is continuous and positive on $(0,x]$; its integral is therefore strictly positive. Integrating the last display proves the required strict inequality for $A(\cdot,x)$. □

Since $\frac{1}{2}\log 2 < 1$, Theorem 1 implies exactly the strict-concavity statement in [1, Conjecture 5.1]. Its strict Turán consequence, for $p > 2$ and $0 < x < 1$, follows from the strict midpoint inequality in Theorem 1, positivity and the arithmetic–geometric mean inequality:

$$A(p,x)^2 > A(p-1,x)A(p+1,x).$$

No optimality of the interval in Theorem 1 is asserted.

Source scope. The 2019 question uses $(1,\infty)$ as its parameter domain. The older [2, Conjecture 2.4] used $(0,\infty)$, whereas its Theorem 2.6 proves separate parameter concavity for the two-parameter integral $\operatorname{arcsinh}_{p,q}$ with one parameter held fixed. The proof above differentiates the diagonal $p = q$ explicitly; separate concavity alone does not justify that step. The binary-entropy estimate is classical and is not presented as a new method.

Formal certificate and review status. The Lean development defines (1) using real powers and a genuine interval integral. Its closed endpoint `ArcsinhPositive.source_conjecture` states the exact $p > 1$ source claim. The stronger endpoint `sourceArcsinh_strictConcaveOn` states Theorem 1 with only $x > 0$ as a hypothesis. The certificate proves the entropy identity, both derivatives, pointwise strict concavity, continuity at $t = 0$, and strictness of the integrated Jensen inequality. It does not require differentiation under the integral sign.

A fresh isolated replay passes all 19 declarations in five modules with warnings treated as errors. Their axiom reports contain only `propext`, `Classical.choice` and `Quot.sound`. The pinned toolchain is Lean 4.34.1, with mathlib revision `d13f23b723b8a846827a245b89c10fc7d3f11612`.

Independent source, mathematical, formal, manuscript and bounded novelty reviews have been completed. The possible connection with a generic reciprocal-norm parameter theorem was screened explicitly; no established subsuming result was located within the recorded search limits. No worldwide priority is claimed. The entire result is one family; supporting identities and consequences are not separate outcomes.

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References

[1] L. Yin, L.-G. Huang, Y.-L. Wang and X.-L. Lin, *A survey for generalized trigonometric and hyperbolic functions*, J. Math. Inequal. **13**(3) (2019), 833–854. doi:10.7153/jmi-2019-13-58.

[2] Á. Baricz, B. A. Bhayo and M. Vuorinen, *Turán type inequalities for generalized inverse trigonometric functions*, Filomat **29**(2) (2015), 303–313. doi:10.2298/FIL1502303B.