# A clique with a five-edge tail reverses the hubs-biased Kirchhoff ordering

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## Status and abstract

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Estrada and Mugnolo conjectured that the hubs-repelling Kirchhoff index dominates the ordinary Kirchhoff index, which in turn dominates the hubs-attracting index [1, Conjecture 4.13]. For the spectral quantity in their Lemma 4.6, the graph consisting of $K_6$ with a pendant path of five edges has $R_1 = 1696684/14035 < 370/3 = R_0$. We give exact rational inverse certificates and a Lean dossier with uniqueness proofs over both the rationals and the reals. The independently reviewed experiment has no spectral failures on all 12,103 connected graphs of orders 5 through 8, consistent with the source's reported computation. Global minimality remains open: the smallest spectral counterexample has order between 9 and 11.

The source's potential, Moore-Penrose, and spectral descriptions are not mutually equivalent on irregular graphs. We explain the distinction without claiming credit for these observations. The spectral counterexample persists after that distinction is made; it is the substantive result reported here.

## 1. Source objects and the precise question

Let G be a finite simple connected unweighted graph on $n \geq 2$ vertices. Write A for its adjacency matrix, $d_i$ for its positive degrees, $K = \mathrm{diag}(d_i)$, and 1 for the all-ones column. The biased Laplacian in [1, (3.8)] is defined by the following entries (we also allow $\alpha = 0$):

$$c\alpha(i) = \Sigma_j A_{ij}(d_i/d_j)^{\wedge}\alpha,\ L\alpha = \mathrm{diag}(c\alpha) - K^{\wedge}\alpha\ A\ K^{\wedge}(-\alpha).$$

Thus $L\alpha 1 = 0$ and $L_0$ is the ordinary combinatorial Laplacian. The source uses $\alpha = 1$ for hubs-repelling and $\alpha = -1$ for hubs-attracting. Definition 4.1 defines resistance as a potential drop divided by the imposed current. Lemma 4.2, equation (4.3), instead specifies a Moore-Penrose matrix expression. For $X = L\alpha^{+}$ it is

$$\Omega\alpha(i,j) = X_{ii} + X_{jj} - X_{ij} - X_{ji};\ R\alpha^{MP} = \Sigma_{i<j}\ \Omega\alpha(i,j).$$

The sum is Definition 4.5, equation (4.15). Both off-diagonal entries are essential, since X need not be symmetric. Lemma 4.6, equation (4.16), asserts the different spectral expression

$$R\alpha^{sp} = n\ \Sigma_{k=2}^{n}\ 1/\rho\alpha,_k = n\ \mathrm{tr}(L\alpha\#).$$

Here $0 = \rho\alpha,_1 < \rho\alpha,_2 \leq \ldots \leq \rho\alpha,_n$ are the eigenvalues and # denotes the group inverse. Section 3 below justifies this spectral interpretation independently of the source's Moore-Penrose argument. Conjecture 4.13, equation (4.25), asserts $R_1 \geq R_0 \geq R_{-1}$, with equality if and only if G is regular. Violating the first inequality already refutes that universal assertion, regardless of the interpretation of its equality clause.

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# 2. Why the three source descriptions separate

## The physical equation has a compatibility obstruction

Set $w_i = d_i^{(-2\alpha)}$. For adjacent i,j the identity $w_i(d_i/d_j)^{\alpha} = (d_i d_j)^{(-\alpha)}$ is symmetric in i,j. Consequently $w^T L_\alpha = 0$. The potential equation in the proof of Lemma 4.2, namely $L_\alpha V = I(e_i - e_j)$ [1, (4.7)], requires $I(w_i - w_j) = 0$. For nonzero I and $\alpha = \pm 1$ this requires equal degrees at the source and sink. An irregular graph has unequal-degree pairs, so this equation does not define an all-pairs physical resistance index.

This is a necessary compatibility condition for the source's equation, rather than a proposed repair of its directional Ohm-law or antisymmetry assumptions. In particular, we do not claim that every possible corrected physical resistance model is refuted by our graph. For the witness below, $w_0 - w_{10} = -35/36$ when $\alpha = 1$, giving an explicit incompatible pair.

## The non-orthogonal similarity does not preserve Penrose inversion

The source diagonalizes $L_\alpha$ using a non-orthogonal change of basis and treats the correspondingly transformed reciprocal-eigenvalue matrix as its Moore-Penrose inverse in the proof of Lemma 4.2; Lemma 4.6 then uses its trace. Similarity preserves the group-inverse equations, but in general it does not preserve the two transpose equations defining the Moore-Penrose inverse. Orthogonal similarity would preserve those equations; the degree scaling used here is not orthogonal on irregular graphs.

Accordingly, (4.3) and Lemma 4.6's formula (4.16) give different numbers. The source's equation numbered (4.6) is the vertex potential-balance equation, not the spectral formula; this numbering distinction matters when describing the inconsistency. For our witness at $\alpha = 1$ the true Moore-Penrose pair sum is 515538204/7137235, whereas the spectral value is 1696684/14035. The dossier contains proofs that the group inverse is not a Penrose inverse at this witness. These observations receive no separate discovery credit.

## The reported computation diagnoses the intended numerical quantity

Before Conjecture 4.13, the authors report computations on more than 12,000 connected graphs with $5 \le n \le 8$. The independent frozen review [R] checked the connected graph datasets exactly and obtained the following counts. No equality-versus-regularity failures were found in either reading.

| Order | Connected graphs | MP failures | Spectral failures |
|---|---|---|---|
| 5 | 21 | 2 | 0 |
| 6 | 112 | 10 | 0 |
| 7 | 853 | 56 | 0 |
| 8 | 11,117 | 545 | 0 |
| Total | 12,103 | 613 | 0 |

At n = 5 the graph $K_4$ with a pendant vertex already has $R_1^{MP} = 33856/4317 < 17/2 = R_0$. The Moore-Penrose reading therefore contradicts the source's reported exhaustive data inside its stated range. By contrast the spectral reading passes every graph in that range. This strongly identifies the computed quantity as spectral; the authors' implementation was not recovered, so this remains an inference about their code. The review also enumerated all connected graphs of orders 2, 3, and 4. Taken together, these checks establish the diagnostic statement that the spectral ordering holds through n = 8. This finite enumeration is supporting computational evidence, not a Lean theorem or a proof of the conjecture.

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# 3. The eleven-vertex counterexample

## Theorem and graph

Let H have vertices 0,…,10. Join every pair in {0,…,5}, and add edges {0,6}, {6,7}, {7,8}, {8,9}, {9,10}. This is $K_6$ with a pendant five-edge path, and it is simple and connected. Its degree sequence is (6,5,5,5,5,5,2,2,2,2,1). Then

$$R_1^{sp}(H) = 1696684/14035 < 370/3 = R_0(H).$$

Hence the spectral reading of Conjecture 4.13 is false. Independently, $R_1^{MP}(H) = 515538204/7137235 < 370/3$, so its literal Moore-Penrose reading is false as well. This is one spectral-ordering counterexample, with a secondary source inconsistency; it is not two claimed results.

## Ordinary proof of the spectral correspondence

Let W = diag(w). The matrix WLα is a symmetric weighted Laplacian whose edge conductances are $(d_i d_j)^{(-\alpha)}$. For any real x, $x^T WL\alpha x$ is the sum over edges of $(d_i d_j)^{(-\alpha)}(x_i - x_j)^2$. It is nonnegative, and connectedness makes its kernel precisely the constants. Since W is positive diagonal, $S = W^{(1/2)}L\alpha W^{(-1/2)} = K^{(-\alpha)}L\alpha K^{\alpha}$ is symmetric positive semidefinite with one-dimensional kernel. Thus Lα is diagonalizable, zero is simple, and every other eigenvalue is positive.

In a diagonalization of Lα, replace the simple zero eigenvalue by zero and every other eigenvalue by its reciprocal. The resulting matrix satisfies LGL = L, GLG = G, and LG = GL. Uniqueness of the group inverse (proved below and in Lean) identifies it with Lα#. Trace is similarity invariant, so $\mathrm{tr}(L\alpha\#) = \Sigma_{k=2}^{n} 1/\rho\alpha,_k$. This argument justifies the spectral reading without using any similarity law for the Moore-Penrose inverse. The eigenvalue argument in this paragraph is ordinary mathematics, not part of the Lean bridge.

## Exact values and the decisive inequality

The rational certificates give the following values; the α = −1 row entries come from the independent exact review, while the formal refutation requires only α = 0 and 1.

| Reading | α = −1 | α = 0 | α = 1 |
|---|---|---|---|
| MP pair sum | 21930927/278132 | 370/3 | 515538204/7137235 |
| Spectral trace | 1096887/11036 | 370/3 | 1696684/14035 |

Cross multiplication gives $R_0 - R_1^{sp} = 102898/42105 > 0$. No floating-point eigenvalue estimate is involved. As an independent check on the ordinary value, each clique pair has resistance 1/3, yielding 5 in total. Among vertices 0,6,…,10 the path contributes 35. For a clique vertex $i \neq 0$ and a path vertex at distance t from 0, resistance is 1/3 + t. The 25 such pairs contribute 5(5/3 + 15) = 250/3. Their sum is 370/3. Section 4 explains how the biased value is certified by finite rational identities.

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# 4. Exact certificates and uniqueness

## A compact description of all matrix entries

The full 11 × 11 rational Laplacians, Moore-Penrose inverses, group inverses, and intermediate products for α = 0,1 appear explicitly in formal/Certificates.lean. The independent review's witness-exact.json also supplies all three α values as arrays of rational strings. The following formulas describe these large matrices without printing hundreds of entries in the paper. They specify every entry once the displayed graph and degree sequence are fixed.

Put $P = 1w^T/(w^T1)$, $U = 11^T/n$, and $T = ww^T/(w^Tw)$. Then define

$$G = (L\alpha + P)^{-1} - P, \quad M = (I - U)G(I - T).$$

The rank-one P is the spectral projection onto ker Lα along im Lα: $P^2 = P$ and $L\alpha P = PL\alpha = 0$. On the zero eigenspace, Lα + P acts as the identity; on every other eigenspace it acts as Lα. It is therefore invertible. This gives LG = GL = I − P and the three group-inverse identities. Since im L is the orthogonal complement of w and ker L = span{1}, projection yields LM = I − T and ML = I − U. Both are orthogonal projections; hence LML = L and MLM = M, with both LM and ML symmetric. This proves all four Penrose conditions for M.

## Finite checking and a sample of the construction

Every construction step can use exact rational Gaussian elimination. For α = 1, w = (1/36,1/25,1/25,1/25,1/25,1/25,1/4,1/4,1/4,1/4,1), and $w^T1 = 401/180$. The first row of $L_1$ is (9,−6/5,−6/5,−6/5,−6/5,−6/5,−3,0,0,0,0); an ordinary clique vertex has diagonal 29/6, entry −5/6 to vertex 0, and four entries −1. Vertex 6 has row entries −1/3 at 0, 4/3 at 6, and −1 at 7. These rules and the edge formula in Section 1 determine $L_1$ completely.

The formal certificate does not trust a Gaussian-elimination program. It checks the resulting matrix equations and index sums in the kernel using rational arithmetic. Separate certificates verify the four Penrose equations for $M_0$,$M_1$ and the three group-inverse equations for $G_0$,$G_1$. The graph module proves that its Laplacians equal the displayed $L_0$,$L_1$ and that the witness is connected. The index equalities then reduce to exact rational identities.

## Why the inverse cannot be chosen differently

Here is an algebraic uniqueness proof, valid over the reals or rationals. If X,Y are Penrose inverses of A, then (AX)(AY) = AY and (AY)(AX) = AX. Transposing the first identity, using symmetry of AX and AY, identifies AX = AY. Similarly the symmetric products XA and YA agree. Thus X = XAX = YAX = YAY = Y. If X,Y are group inverses, commutation gives AX = (AYA)X = Y(AXA) = YA = AY; then X = XAX = XAY = AXY = AYY = YAY = Y. This removes all inverse-choice ambiguity. In particular, existential inverse data in the conjecture cannot avoid the certified strict inequality. Both uniqueness arguments are formalized over the rationals in Basic.lean and repeated over the reals in RealBridge.lean.

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# 5. What the Lean dossier establishes

## Correspondence between source language and declarations

All five supplied modules, including Audit.lean, are byte-identical to kirchhoff-formal-r2/formal. The following table summarizes the primary real endpoints; statement-correspondence.md records the fuller graph-level mapping.

| Object | Lean declaration and scope |
|---|---|
| Real MP refutation | KirchhoffReal.refuteMP: negates LiteralConjecture Penrose pairIndex. |
| Real spectral refutation | KirchhoffReal.refuteSpec: negates LiteralConjecture GroupInverse spectralIndex. |
| Quantifiers | LiteralConjecture: every $n \geq 2$, connected graph, and $\alpha = \pm 1$; existential inverse data, ordering, and regularity iff equality. |
| Inverse equations | Penrose: AXA=A, XAX=X, $(AX)^T$=AX, $(XA)^T$=XA. GroupInverse: AXA=A, XAX=X, AX=XA. |
| Laplacian | Kirchhoff.laplacian; laplacian_source_formula identifies (3.8). |
| MP index | pairIndex: sum of $X_{ii} + X_{jj} - X_{ij} - X_{ji}$ over $i < j$. |
| Spectral index | spectralIndex: n tr(X); ordinary eigenvalue bridge in Section 3. |

The real Laplacian is the entrywise rational-to-real scalar extension of the graph Laplacian. The bridge proves preservation of matrix products, transpose equations, pair sums, and the trace index. The final proofs specialize the universal existential statement to $n = 11$, the connected witness, and $\alpha = 1$; uniqueness replaces the existential matrices with the certified ones, after which exact arithmetic contradicts the ordering. They do not rely on the selectors' zero defaults outside inverse existence predicates. No global inverse-existence theorem is assumed: existence is explicit in the conjecture formulation, and proved at the witness.

## Replay, audit, and negative controls

The portable source-only packet contains no compiled local modules. Set LEAN_BIN to the Lean executable and LEAN_PACKAGES to the read-only package cache, then run python3 -I replay.py after safe extraction into an empty directory. The required Lean version is 4.34.1, commit 5045d0056413266e57c625dcd7c365b10e377c52; mathlib is pinned to d13f23b723b8a846827a245b89c10fc7d3f11612. The package manifest is included. These external dependencies are not vendored.

The driver refuses existing oleans and verifies its source hashes. It builds sequentially with -j1 -DwarningAsError=true and runs leanchecker per module. Every Lean or checker call receives at most two CPU affinities, a 120-second CPU limit, a 140-second wall limit, MALLOC_ARENA_MAX=2, and a sampled 6 GiB aggregate process-group RSS watchdog. It records full logs and structured receipts. The full audit selects every declaration from the four proof modules by module identity, including generated declarations, and records complete type and opaque-access body ASTs, safety flags, and transitive axioms. There are 176 local declarations.

The only permitted axioms are propext, Classical.choice, and Quot.sound. No new axiom, sorry, admit, unsafe definition, native_decide, or ofReduceBool is used. Three altered false claims are required to fail. Because the predecessor's rejection attempts can fail at elaboration rather than on the intended semantic ground, the replay also checks the reviewer's positive proofs of the corresponding negations: the group inverse is not MP, an entry perturbation is not MP, and an old incorrect numerical value is false.

## Trust boundary

The Lean endpoints concern the group-inverse trace and the literal Moore-Penrose pair sum. The eigenvalue-to-trace identification is proved ordinarily in Section 3, not formalized in the unchanged modules. The physical obstruction, exhaustive small-graph experiment, minimality bounds, and literature assessment are also outside the Lean endpoints. The audit still trusts Lean's kernel and implementation, the pinned cached dependencies, and the operating system; the receipt and hash machinery is not a theorem about its own correctness. A successful replay does not establish novelty or root acceptance.

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# 6. Minimality, related work, and access limits

## Minimality and scope

The independent exact review [R] enumerated connected graphs through order 8 and found no spectral ordering violation. The witness gives an upper bound of 11, so the smallest spectral counterexample order lies in [9,11]. Orders 9 and 10 were not exhaustively tested. Within the clique-plus-pendant family $K_k$ with a path of n−k edges ($2 \le k < n$), the review searched $3 \le n \le 11$; its first failure is n = 11, k = 6. This establishes the first failing order in that family, not global minimality. The graph datasets' exhaustiveness at orders 5–8 is inherited from their published coverage and the review's parser/count checks, rather than formally certified enumeration.

Only the universal ordering is refuted. No replacement classification, extremal bound, asymptotic claim, or statement about arbitrary repaired physical resistance is established. The equality condition need not be used. In particular, the secondary MP defect within the authors' computational range receives no separate credit.

## Related sources and the bounded prior-art record

The source bodies supporting the target statement are the cached arXiv v2 and the author-hosted published JMAA article [1], read in the independent review. We rechecked the relevant v2 statements and proofs. Miranda and Estrada's advection-diffusion paper [2] studies the adjoint degree-biased operator. Its Proposition 3.9 gives the stationary vector d^(−2α), and Theorem 3.11 gives convergence to its span; these are compatible with the left-null-vector calculation here. Its verbal bias signs should not be silently imported into [1]. The frozen review read its relevant body and found no resolution of this ordering.

The citing book by van Gennip and Budd [3] discusses hubs-biased resistance in its literature overview; the inherited review screened its text and read the relevant paragraph and bibliography. It is not a book by Mugnolo, despite the inherited local basename. The Chaos 33 (2023) paper [4] concerns biased Kuramoto dynamics; only its beginning through part of the analytical section was read in the earlier review. Signed-redirect access failures prevented full-body screening. It cannot be used as a fully cleared citation.

Miranda Barrado's 2026 doctoral thesis [5] was subsequently retrieved and text-screened by root (344 pages). The supplied root receipt records no occurrence of 'Kirchhoff'; resistance occurs in the bibliography, and [1] is cited for biased Laplacians and spectral bounds at printed pages 75, 81, 83, 94, and 190. Root reported no Kirchhoff-ordering discussion. This lane retains that evidence and receipt; it does not claim a new full visual reading of all 344 pages. Root also supplied a later current-registry screen and a qualified eligibility ruling. Those bounded checks are not an exhaustive literature or erratum search, and no worldwide priority claim follows. No new external search or access retry was made in preparing this manuscript.

## References and evidence


[1] E. Estrada and D. Mugnolo, Hubs-biased resistance distances on graphs and networks. J. Math. Anal. Appl. 507 (2022), 125728. DOI: 10.1016/j.jmaa.2021.125728. arXiv:2101.07103v2. Relevant body statements: (3.8), Definitions 4.1 and 4.5, Lemmas 4.2 and 4.6, Conjecture 4.13.

[2] M. Miranda and E. Estrada, Degree-biased advection-diffusion on undirected graphs/networks. Math. Model. Nat. Phenom. DOI: 10.1051/mmnp/2022034. Author manuscript retained; body-reading scope as above.

[3] Y. van Gennip and J. Budd, A Prolegomenon to Differential Equations and Variational Methods on Graphs. Cambridge University Press; cached Cambridge body. Relevant overview and bibliography read in [R].

[4] Topologically induced suppression of explosive synchronization. Chaos 33 (2023). DOI: 10.1063/5.0142418. Partially read only; not a full-body clearance.

[5] M. Miranda Barrado, Diffusion operators on graphs and its applications. Doctoral thesis, Universitat de les Illes Balears, 2026. Institute-hosted body and root retrieval receipt retained.

[R] Internal independent review, kirchhoff-review-r1, REVIEW.md and review.json; later root adjudication and root-prior-art.json. These are unpublished computational and access records, not external publications. All source copies, exact matrices, raw MathDB line, and retrieval receipts are hash-bound in the dossier.


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