An adverse non-Q graph from the cyclic Latin square of order six

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Abstract. Join two distinct cells of the cyclic Latin square of order six when they share a row, column or symbol. We prove that this 36-vertex graph has stability number 5 and convex quadratic stability number 6, with least adjacency eigenvalue −3. The latter two invariants remain unchanged after deletion of the open neighbourhood of every vertex. Thus the graph is adverse but is not a Q-graph, refuting Conjecture 4.2 in Cardoso's survey [1], also under the non-completeness condition in the 2016 author manuscript [2]. We give an ordinary proof with explicit rational witnesses and describe a hash-bound Lean certificate. Literature coverage is incomplete; in particular Luz's 2020 paper is unread.

1. Definitions and the source conjecture

Let G be a finite simple undirected graph with at least one edge, A its real adjacency matrix, $\lambda_{\min}(G)$ its least eigenvalue, and $\alpha(G)$ the largest cardinality of an independent vertex set. Put $\tau = -\lambda_{\min}(G) > 0$ and $H = A/\tau$. Equation (11) of [1] defines the optimization problem P(G) and its value $\upsilon(G)$ by

$$P(G):\quad \upsilon(G) = \max\{2\cdot 1^{\mathsf T}x - x^{\mathsf T}(H + I)x : x \ge 0\}. \qquad (1)$$

The inequalities on x are coordinatewise. The matrix H + I is positive semidefinite, so the displayed objective is concave. The sentence in [1] calling H itself positive semidefinite is inconsistent with its negative eigenvalue; we follow the displayed formula, as does [2]. A Q-graph (or convex-QP graph) is a graph with $\alpha(G) = \upsilon(G)$.

Write $N(v) = \{w : vw \text{ is an edge}\}$. The notation $G - N(v)$ means the induced graph on $V(G) \setminus N(v)$: v remains. In §4 of [1], G is **adverse** if it has no isolated vertices, both $\upsilon(G)$ and $\lambda_{\min}(G)$ are integers, and for every v,

$$\upsilon(G - N(v)) = \upsilon(G), \quad \lambda_{\min}(G - N(v)) = \lambda_{\min}(G). \qquad (2)$$

No condition excludes isolated vertices in the deletion graphs. In fact the retained vertex v is isolated. The abstract of [2] additionally says "non complete"; its Definition 2 does not repeat that adjective. Our witness satisfies either reading.

History. The survey attributes the adverse concept to the 2001 report [3] and Conjecture 4.2 to its reference [10], namely [2]: all adverse graphs are Q-graphs. The EURO 2012 abstract [4] already assumes this assertion in an algorithmic application. It remains explicitly open in [2] and in the 2018 arXiv survey, published in 2020 [1]. These are dated source statements, not a claim that the literature remains clear today.

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2. The graph and an attained least eigenvalue

Let $V = (\mathbb{Z}/6\mathbb{Z})^2$. For distinct $p = (i,j)$ and $q = (i',j')$, declare p adjacent to q exactly when $i = i'$, $j = j'$, or $i + j = i' + j'$. The 18 lines are the six rows, six columns and six symbol classes. Each has six cells, and each cell lies on three lines. Distinct cells share at most one line. Thus the degree is $3 \cdot 5 = 15$; in particular there are no isolated vertices. Row and column moves also show connectedness.

Let B be the 18 by 36 zero-one line-incidence matrix. Counting shared lines gives the exact identity

$$A = B^{\mathsf{T}}B - 3I. \tag{3}$$

Translation $T_{a,b}(i,j) = (i+a,j+b)$ permutes all three line families and is a graph automorphism. Consequently it suffices in the ordinary proof to treat deletion at $o = (0,0)$. At this base the retained cells are o together with cells satisfying $i \neq 0$, $j \neq 0$ and $i+j \neq 0$. There are $1 + 25 - 5 = 21$ such cells.

For any induced vertex set S, restrict B to its columns S, obtaining B_S. Its adjacency matrix is $B_S^{\mathsf{T}}B_S - 3I$. Therefore for every real z on S,

$$z^{\mathsf{T}}A_S z = \|B_S z\|^2 - 3\|z\|^2 \geq -3\|z\|^2. \tag{4}$$

Every adjacency eigenvalue is at least -3, globally and after each deletion. To prove attainment, take the following matrix of cell weights z at o (row i, column j, ordered 0 through 5):

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Each row and column sum is zero. The only nonzero symbol classes are 3, 4 and 5, each with one $+1$ and one -1. Hence $Bz = 0$. Every nonzero entry has nonzero row, column and symbol, so z vanishes on $N(o)$, and $z \neq 0$. Equation (3) gives $Az = -3z$; restricting to the deletion gives the same eigen-equation. At any $v = (a,b)$, use $z_v(i,j) = z(i-a,j-b)$. Translation preserves the kernel equations and the required support. Thus

$$\lambda_{\min}(G) = \lambda_{\min}(G - N(v)) = -3 \quad \text{for every } v. \tag{5}$$

As a separate check on existence, the deletion matrix B_S has 21 columns and 18 rows, hence a nonzero kernel. The explicit vector above is what connects the ordinary argument to the finite rational Lean certificates; dimension counting is not needed by that certificate.

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3. The quadratic value and the stability number

Fix S = V or S = V \ N(v), and put y = B_Sx. Since each retained column has three ones, $\sum_\ell y_\ell = 3\sum_p x_p$. Using (3) and (5), the objective in (1) is

$$2\Sigma x - x^{\mathsf T}(A_s/3 + I)x = (1/3)\Sigma\ell(2y\ell - y\ell^2)$$
$$= 6 - (1/3)\Sigma\ell(y\ell - 1)^2 \le 6. \qquad (6)$$

This bound holds for all real x. Globally x ≡ 1/6 is nonnegative and gives every line sum one, so it attains 6. On the deletion at o, the following explicit fractional transversal is the one used by the formal packet:

1	0	0	0	0	0
0	1/4	0	3/4	0	0
0	1/4	0	0	0	3/4
0	1/4	3/4	0	0	0
0	1/4	0	0	1/2	1/4
0	0	1/4	1/4	1/2	0

It is nonnegative, vanishes on N(o), and its row, column and cyclic-symbol sums are all one. For example, symbol sums for residues 0 through 5 are respectively 1, 3/4+1/4, 1/4+1/2+1/4, 1/4+1/4+1/2, 3/4+1/4, and 3/4+1/4. Translate the weights as in §2 for every base v. The restricted vectors attain equality in (6). Consequently

$$\upsilon(G) = \upsilon(G - N(v)) = 6 \quad \text{for every } v. \qquad (7)$$

There is also a simpler ordinary fractional witness: give weight 1 to o, weight 1/4 to every other retained cell, and zero to deleted cells. A line through o retains only o. Any other row, column or symbol line retains four cells, because its two forbidden intersections are distinct. Each line therefore sums to one. Translation again works. This alternative is explanatory; the unchanged Lean modules use the displayed rational matrix.

For a regular graph the survey identifies υ with the Hoffman bound. Here this gives 36·3/(15+3) = 6. That classical global formula alone does not prove (7): the deletion graph is irregular and has an isolated vertex.

Stability. In an independent set all rows, columns and symbols are distinct, so its size is at most six. If it had size six, each of the three coordinate lists would be a permutation of $\mathbb{Z}/6\mathbb{Z}$. The row and column residue sums are both 0+1+2+3+4+5 = 3. Their sum is 0 modulo 6. But the symbol sum must be 3, contradicting symbol = row + column. This is the classical even-cyclic transversal parity obstruction. The five cells (0,0), (1,1), (2,2), (3,4), (4,5) have distinct rows, columns and symbols (0,2,4,1,3). Hence α(G) = 5.

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4. Refutation, robustness and scope

Theorem. The cyclic Latin square graph of order six is adverse and is not a Q-graph. It refutes the universal assertion in Conjecture 4.2 of [1].

Proof. The graph is finite, simple and 15-regular. Equations (5) and (7) give integral least eigenvalue and quadratic value, unchanged after every open-neighbourhood deletion. These are precisely the adverse conditions (2). Section 3 proves $\alpha(G) = 5 < 6 = \upsilon(G)$, contradicting the defining equality for Q. **.**

The witness has an edge, for example (0,0)(0,1), satisfying the ambient assumption of the survey. It is noncomplete: (0,0) and (1,1) are distinct and nonadjacent (symbols 0 and 2). Thus imposing “non complete” as in the abstract of [2] does not rescue the assertion. In the earlier report [3], adverse subgraphs are defined relative to a parent graph, with additional equalities to the parent invariants; taking the parent to be G itself makes those equalities identities. This observation concerns the inspected author report, not all unavailable versions.

The retained base vertex is isolated in every deletion. Removing a closed neighbourhood would be a different condition and is not what these sources define. Our conclusion relies on open deletion throughout; it does not change that convention.

Why might the computations have missed this graph?

Table 1 of [2] reports Johnson graphs and complements, complements of Hamming graphs, the Petersen graph and iterated line-graph examples. Every listed graph is Q. The cyclic Latin square graph of order six is not named in that table. A search focused on these benchmarks can therefore miss this 36-vertex example. This is a possible explanation from the reported sample, not a reconstruction of all experiments, an accusation of a computational error, or a claim that the authors never considered Latin squares.

A mechanism, not a classification

The example combines a nonattained integer Hoffman bound with preservation of the quadratic optimum under open-neighbourhood deletion. Latin square graphs belong to the strongly regular setting, making such graphs natural candidates. More generally, if a regular graph has integral least eigenvalue and integral υ, no isolated vertices, $\alpha < \upsilon$, and both invariants survive every open deletion, then it is an adverse non-Q graph simply by the definitions. Strong regularity alone does not imply the deletion equalities or nonattainment. We assert no classification, no result for every Latin square, and no additional credit for parameter variants.

Here the incidence factorization separates the two deletion tasks: a supported kernel vector fixes the least eigenvalue, while a supported nonnegative vector with all line sums one fixes the optimum. Both are necessary parts of the argument; parity only supplies the final failure of integrality. These classical ingredients are not individually claimed as discoveries.

Ordinary proof versus formal proof

The ordinary proof uses translation to reduce all bases to one. The formal packet verifies exact rational certificates at all 36 bases in six deletion modules and transports the equations to real matrices. Both establish the same quantified deletion conclusions. Connectedness, the alternate constant fractional witness, dimension counting and this contextual discussion are explanatory observations, not extra claims of Lean coverage.

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5. Formal statement correspondence

All declarations in the left dossier namespace below are in Adverse381433 unless prefixed IndependentReview. The 13 proof modules are copied byte-for-byte from adverse-formal-r2. Reviewer.lean is copied byte-for-byte from the independent reviewer and is explicitly reviewer-authored. Audit.lean is retained unchanged as infrastructure; DossierAudit.lean extends the audit to Reviewer.

Source object / endpoint	Lean definitions / theorems
Cells; adjacency; real matrix	Cell; latinGraph; adjacency (Definitions)
Least eigenvalue, nonzero real eigenvectors	realEigenvalues; leastEigenvalue (sInf)
Equation (11): P(G), υ, x ≥ 0	objective; upsilon (sSup of feasible values)
Independence and largest cardinality α	Independent; alpha; independent_card_le_five; independentFive_valid; alpha_eq_five
Open neighbourhood deletion	deletionVertices; deleteOpen (base retained)
Adverse; Q equality	Adverse; QGraph
Incidence Gram identity over rationals and reals	incidence_identityQ; incidence_identityR; deletion_gram
Translation, rational support and line sums	translate_adj; CertificateAt; certificate_i_j (all 36 pairs); all_certificates
Least eigenvalue lower bound, attainment	gram_eigenvalue_lower; leastEigenvalue_eq_of_gram; kernel_eigen
λmin = −3 globally and for all deletions	global_least; deletion_kernel_sum; deletion_least
Quadratic line-sum bound and attainment	objective_gram; objective_le_six; upsilon_eq_six
υ = 6 globally and for all deletions	global_upsilon; deletion_fractional_sum; deletion_upsilon
No isolated vertices; adverse, non-Q	no_isolated_vertices; graph_adverse; graph_not_Q
Universal conjecture with edge hypothesis	SourceConjecture; refutation_of_source_statement
Universal version without edge hypothesis	CardosoLuzConjecture; refutation
Noncomplete stricter source (reviewer)	IndependentReview.NonComplete; IndependentReview.SourceRestatement; IndependentReview.noncomplete; IndependentReview.refutes_stricter_source

The sInf/sSup definitions are bridged to attained values on the witness and each deletion: eigenvalue sets are nonempty and bounded below, and feasible objective values are nonempty, bounded above and attain 6. Thus no empty-set default or unbounded supremum supplies the conclusion. The stability maximum is likewise attained. The denominator is 3. Conventions for unrelated edge-free graphs play no role.

SourceConjecture ranges over finite vertex types in Type and includes an edge. This universe contains the counterexample, which suffices to refute the source’s universal statement. The reviewer endpoint separately spells out every adverse condition together with an edge and a distinct nonedge; it is not merely a relabelling of the main conclusion.

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6. Reproducibility, trust and literature limits

The source-only ZIP has a manifest, immutable proof sources, audit infrastructure, portable replay driver and negative controls. A fresh safe extraction rejects path traversal, absolute names, symlinks and compiled artifacts. The replay refuses existing oleans, uses LEAN_BIN and LEAN_PACKAGES, and builds sequentially with lean -j1 -DwarningAsError=true. Each proof module and the reviewer module is checked by leanchecker. A full declaration audit prints types, bodies and transitive axioms for all 167 declarations, including generated declarations. Six mathematical/certificate negative controls must fail; a seventh checks freshness refusal. Receipts and the compressed full audit accompany the full dossier.

Pins: Lean 4.34.1, commit 5045d0056413266e57c625dcd7c365b10e377c52; mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612. Every Lean invocation, including checker, version and controls, is watched: at most two CPUs, CPU 120 s, wall 140 s, total process-group RSS sampled every 0.1 s with a 6 GiB kill threshold, and MALLOC_ARENA_MAX=2. Only one heavy process runs at a time. Sampling is not an instantaneous memory bound.

Only propext, Classical.choice and Quot.sound are permitted axioms. No sorry, admit, new axioms, unsafe, native_decide or ofReduceBool is used in the proof modules. Finite rational checks use kernel reduction. Trust remains in the Lean kernel/toolchain, the supplied mathlib environment, hardware and the correspondence between source definitions and the formal model. Cached dependency oleans are not rebuilt from first principles. A kernel proof does not certify historical accuracy, novelty or significance.

Access limits and related work. This manuscript uses the hash-bound survey [1], the author manuscript [2], targeted readable passages of the imperfectly extracted report [3], and the EURO abstract [4]. The 1995 bound and the regular-graph Hoffman formula are described through [1], not independently reread originals. Formal verification settles this counterexample’s mathematics, not its priority. The inherited review held novelty and significance; the root later issued QUALIFIED_PASS for preparation and final review, with zero accepted credit. We do not upgrade that administrative ruling into literature clearance.

Luz 2020 is unread. C. J. Luz, “New results for recognizing convex-QP adverse graphs,” Optimization 69(1) (2020), 131–144, DOI 10.1080/02331934.2018.1556664 (online 2019), is described here only at abstract level: recognition criteria for convex-QP adverse graphs. Its body was not obtained. Root noted that it shares a special issue with the survey; co-publication is context, not proof of nonsubsumption. Other gaps include final-version equivalence of author reports, inaccessible earlier recognition papers, the corrupted Cardoso 2003 extraction, the editorial body, unidentified citing works and complete current proof-registry content. No new live literature search was undertaken for this draft. No priority claim is made.

References (bodies or passages actually read)

[1] D. M. Cardoso, A survey on graphs with convex quadratic stability number, arXiv:1811.05516 (2018); Optimization 69(1) (2020), 1–22. DOI 10.1080/02331934.2018.1526282. Cited body: cached arXiv PDF/text, especially equation (11), §4 and bibliography.

[2] D. M. Cardoso and C. J. Luz, A simplex like approach based on star sets for recognizing convex-QP adverse graphs, J. Combinatorial Optimization 31(1) (2016), 311–326. DOI 10.1007/s10878-014-9745-x. Cited body: 16-page author manuscript, abstract, Definition 2, §4, Table 1 and conclusions; final-version identity not independently established.

[3] D. M. Cardoso and C. J. Luz, Extensions of the Motzkin-Straus Result on the Stability Number of Graphs, Cadernos de Matemática CM01/I-17 (2001). Cited body: author-hosted revised PostScript, locally converted, 21 pages; targeted definitions and conclusions, imperfect text. Survey bibliography describes 18 pages.

[4] M. F. Pacheco, D. Cardoso and C. J. Luz, EURO XXV (Vilnius, 8–11 July 2012), Programme and Abstracts, printed p. 200, “Convex quadratic programming applied to the stability number of a graph”. Cited body: the abstract in the cached full conference book; not a full research paper.

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