A wedge-complement domain with a nonstarlike Cassinian ball

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Abstract

A thin closed wedge removed from the complex plane gives an open, connected, proper domain U containing 0. It is starlike about 0 both in the segment sense and in the finite-boundary-point strict sense. Nevertheless its strict Cassinian ball of radius 7/4 contains y = 4+i and omits z = y/2. We give rational estimates sufficient for this conclusion and an exact evaluation at y on each of the two boundary rays. Two immutable Lean modules certify the geometric construction and the negations of the universal claims for the two domain hypotheses. The sharper evaluation and motivating slit computation are ordinary mathematical supplements. A bounded literature check found no explicit antecedent in the inspected bodies, but several important bodies remain unavailable; novelty is not established.

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1. Definitions and the assertion being tested

Throughout, C is the Euclidean plane with modulus |.|, and a domain is a nonempty connected open subset of C. For a proper domain D and finite a,b in D, define

c_D(a,b) = sup { |a-b| / (|a-p| |b-p|) : p in boundary D },
B_c,D(a,r) = { b in D : c_D(a,b) < r }, r > 0. (1)

Boundary means the finite Euclidean frontier. Distances in (1) are Euclidean, not chordal. For the witnesses used below the boundary is nonempty, and the ratio set is bounded. The one-point compactification convention in the source adds infinity to an unbounded domain's boundary. For finite a,b its contribution is 0: the quotient tends to 0 as |p| tends to infinity. Adjoining 0 does not change a nonempty bounded supremum of nonnegative numbers. The formal certificate proves this insertion-of-zero fact, rather than constructing the compactification itself.

A set D containing a is ordinarily starlike about a if (1-t)a+tb belongs to D for every b in D and 0 <= t <= 1. Here 'strictly starlike' means that (1-t)a+tp belongs to D for every finite p in boundary D and 0 <= t < 1. This is the boundary-point convention of [2, Section 2]. Another convention requires boundedness and exactly one boundary intersection on every ray; it is not the convention asserted for our unbounded U. The distinction matters: a slit complement is ordinarily starlike but fails the boundary-point condition.

The assertion in [1, Conjecture 4.7] says: for every proper starlike domain D, every starlike center a in D, and every r > 0, B_c,D(a,r) is starlike about a. We test both the ordinary and the boundary-point strict domain hypothesis, retaining the ordinary conclusion. Failure of the ordinary conclusion also prevents the usual stronger strict conclusion for an open ball. A planar counterexample refutes an assertion posed for all dimensions n >= 2; no transport to every dimension is needed.

2. The wedge and its actual frontier

Put

$$K = \{ u+iv : u \geq 1, |v| \leq (u-1)/20 \}, \quad U = C \setminus K. \tag{2}$$

$$y = 4+i, \quad z = 2+i/2, \quad r = 7/4. \tag{3}$$

Theorem. U is an open connected proper domain containing 0. It satisfies both starlikeness conditions of Section 1, but $B_{c,U}(0,7/4)$ is not starlike about 0.

Proof of the domain assertions. K is the intersection of the three closed half-planes $u \geq 1$, $20v \leq u-1$ and $-20v \leq u-1$, so U is open. It contains 0 and excludes 1. Suppose w is in U and bw is in K, where $0 \leq b \leq 1$. Then $b\,\mathrm{Re}(w) \geq 1$ implies $b > 0$. Also

$$20b\,|\mathrm{Im}(w)| \leq b\,\mathrm{Re}(w)-1 \leq b(\mathrm{Re}(w)-1).$$

Division by b shows w is in K, a contradiction. Thus [0,w] lies in U. These segments connect each point to 0, proving path connectedness and hence connectedness. Direct substitution shows y,z are in U.

We claim that the finite frontier is exactly

$$\text{boundary } U = \{ 1+t(1+i/20) : t \geq 0 \} \cup \{ 1+t(1-i/20) : t \geq 0 \}. \tag{4}$$

A frontier point belongs to K since K is closed. If $u > 1$ and both oblique inequalities are strict, the point is interior to K and cannot be in the frontier. At $u = 1$ the inequalities force $v = 0$. The remaining points satisfy $u \geq 1$ and $20v = u-1$ or $20v = -(u-1)$, precisely (4). Conversely, each such point p belongs to K, and p-delta belongs to U for every positive real delta: either its real part is below 1, or an oblique inequality fails. Taking delta down to 0 proves frontier membership. This verifies both inclusions for the actual frontier, rather than substituting a convenient boundary subset.

For strict starlikeness, let p be on a ray in (4) and let $0 \leq b < 1$. The ray equality gives $20|\mathrm{Im}(bp)| = b(\mathrm{Re}(p)-1)$. If bp were in K, it would also give $20|\mathrm{Im}(bp)| \leq b\,\mathrm{Re}(p)-1$, forcing $b \geq 1$. Thus bp belongs to U, including the case $p = 1$. This proves the finite-boundary-point strict condition.

It remains to show that y belongs to the ball while its midpoint z does not. The next two sections do this with a uniform margin. All domain and frontier arguments above occur in Wedge.lean. In particular, the proof does not rely on a numerical picture of the removed wedge.

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3. Bounds and exact ray suprema at y

For p = s+iv on either boundary ray, s >= 1 and 20v <= s-1. The denominator squared in (1) at a = 0 and b = y is

$$F(s,v) = (s^2+v^2)((4-s)^2+(1-v)^2). \quad (5)$$

The following elementary lower bounds are sufficient for the formal proof. Each row applies to its closed interval, so overlaps at endpoints cause no gap.

s interval	lower bound for F(s,v)
[1,5/4]	121/16
[5/4,2]	(25/16)(4+(19/20)²)
[2,3]	4(1+(9/10)²)
[3,4]	9(17/20)²
[4,5]	16(4/5)²
[5,infinity)	25

For example, on [2,3], $s^2 >= 4$, $(4-s)^2 >= 1$, and v <= 1/10 implies $(1-v)^2 >= (9/10)^2$. The other finite intervals use the same factorwise argument. On the final interval use $s^2 >= 25$ and $(4-s)^2 >= 1$. Every displayed lower bound exceeds 153/25. Since $|y|^2 = 17$, every boundary ratio is at most 5/3. The ratio set is nonempty (p = 1), so

$$c_U(0,y) <= 5/3 < 7/4. \quad (6)$$

This uniform bound justifies passing to the supremum. Pointwise strict inequalities without a uniform margin would not justify (6).

Ordinary exact supplement. In fact, on each ray separately the supremum is sqrt(17/10), attained at the apex p = 1. Here is an exact positivity certificate avoiding numerical root selection. For e = +1 or -1 and p = 1+t(1+ei/20), write $F_e(t) = |p|^2 |y-p|^2$. Expansion gives

$$F_+(t)-10 = t\,Q(t)/160000,$$
$$Q(t) = 160801t^3 - 657640t^2 - 187600t + 2224000,$$
$$F_-(t)-F_+(t) = t(32080t^2+64000t+32000)/160000. \quad (7)$$

To check Q > 0, its cubic Bernstein coefficients on [0,2], [2,3], [3,4] are, respectively,

(2224000, 6296800/3, 3291040/3, 504648),
(504648, 625396/3, 44014/3, 84067),
(84067, 460388/3, 486048, 1242624).

These mean Q(a+(b-a)u) = sum from j=0 to 3 of B_j binom(3,j)u^j(1-u)^(3-j). All coefficients are positive. For t = 4+h, h >= 0, $Q = 160801h^3+1271972h^2+2269728h+1242624 > 0$. Thus (7) proves $F_e(t) >= 10$, with equality exactly at t = 0. The apex quotient is $|y|/(|1||y-1|) = sqrt(17)/sqrt(10)$. This sharper equality is not an endpoint claimed by the Lean modules; their bound (6) suffices.

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4. Excluding the midpoint; the slit limit

For a valid use of a boundary witness in a real supremum, first check boundedness at z. On the frontier, $|p| >= 1$. If $s <= 3$, then $v <= 1/10$, so $|z-p| >= |1/2-v| >= 2/5$. If $s > 3$, the real distance $|2-s|$ exceeds 1. Thus all ratios at z are bounded above by $(5/2)|z|$. The point

$$p_0 = 2+i/20$$

lies on the upper ray, with

$$|p_0|^2 = 1601/400,\ |z-p_0| = 9/20,\ |z|^2 = 17/4.$$

Its boundary ratio has square

$$|z|^2 / (|p_0|^2\ |z-p_0|^2) = 680000/129681 > 49/16. \quad (8)$$

The final inequality is the integer inequality $10880000 > 6354369$. All quantities are positive, hence the ratio itself exceeds 7/4. Boundedness and the definition of supremum give $c_U(0,z) > 7/4$. Consequently y belongs to the strict ball and $z = y/2$ does not. The ordinary starlikeness conclusion fails. Together with Section 2 this proves the theorem under both domain hypotheses.

Slit motivation (ordinary supplement). Let G = C minus [1,infinity). This is the zero-angle version of (2); it is open and ordinarily starlike about 0. Its frontier is [1,infinity). A point 2 of that frontier has midpoint 1 outside G, so G fails the strict boundary-point condition. This explains why a positive wedge angle is useful.

For b = y or z and real $p = s >= 1$, minimize $s^2\ |b-s|^2$. At y the polynomial is $s^2((s-4)^2+1)$. Its derivative is $2s(2s^2-12s+17)$, with critical points $3 +/- \sqrt{2}/2$. On [1,infinity), the minimum is 10 at s = 1: the other candidate minimum has value $71/4-2\sqrt{2} > 10$, since $31 > 8\sqrt{2}$. Thus $c_G(0,y) = \sqrt{17/10}$.

At z the polynomial is $s^2((s-2)^2+1/4)$. Its derivative is $s(4s^2-12s+17/2)$; the interior local minimum is at $s = (6+\sqrt{2})/4$. Its value is $(71-8\sqrt{2})/64$, which is less than the endpoint value 5/4 because $71-8\sqrt{2} < 80$. Therefore

$$c_G(0,z) = 4\sqrt{17}/\sqrt{71-8\sqrt{2}} > 7/4. \quad (9)$$

The exact slit evaluations in this paragraph were independently checked in the frozen review. The original Cassinian.lean certifies sufficient inequalities for the slit, rather than these exact optimizer formulas. Equations (7) and (9) are ordinary supplements, not additional formalized theorems. No bounded-domain extension is claimed in this draft.

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5. Formal correspondence and reproducibility

All local names below have prefix Cassinian331705. The module Cassinian.lean is the original slit certificate and is byte-identical to the strengthened frozen lane. Wedge.lean imports it and is also byte-identical to that lane. Sanity.lean supplies explicit instantiations, including the compactification convention. The exact source bytes, raw MathDB line, upstream pins and resource receipts accompany this draft.

Paper content	Lean declaration or group
Definition (1)	cassinian, ball (Cassinian.lean)
Ordinary universal assertion	SourceClaim (Cassinian.lean)
Slit frontier and starlikeness	frontier_G, star (Cassinian.lean)
Slit refutation	source_refutation (Cassinian.lean)
Adjoining the zero contribution	infinity_contribution (Cassinian.lean)
Strict hypothesis and assertion	StrictlyStarlike, SourceClaimStrict (Wedge.lean)
Wedge (2) and actual frontier	Wedge.K_abs, Wedge.frontier_eq
Domain hypotheses	Wedge.open_U, connected_U, proper_U, zero_U, star_U, strict_U
Upper bound in (6)	Wedge.polynomial, ratio_y, y_ball
Boundedness and exclusion (8)	Wedge.bounded_z_wedge, z_not_ball
Ordinary conclusion fails	Wedge.ball_not_star
Both universal negations	Wedge.ordinary_refutation, Wedge.strict_refutation
Convention diagnostic	Wedge.slit_not_strict

The strict endpoint negates the assertion with a strict DOMAIN hypothesis and an ordinary BALL conclusion. This already refutes the asserted ordinary ball starlikeness under that stronger domain assumption; it is not a theorem about every definition called 'strictly starlike' in the literature. The complex plane uses the usual Euclidean norm. All frontier bridges are proved with mathlib's actual frontier.

The kernel-checked argument uses exact rational inequalities and squared norms, not decimal extrema or an external optimization oracle. It establishes nonempty bounded ratio images before using the order-theoretic lemmas for real sSup. The more precise apex optimization in Section 3 is separate from the Lean trust claim. No claim that the formal files prove every sentence of the related-work discussion or every supplementary calculation is made.

Replay uses Lean 4.34.1, commit 5045d0056413266e57c625dcd7c365b10e377c52, and mathlib revision d13f23b723b8a846827a245b89c10fc7d3f11612. The compiler, checker and pinned cached upstream oleans are trusted; upstream is not rebuilt. A fresh safe extraction compiles Cassinian, Wedge and Sanity using -j1 -DwarningAsError=true, and runs leanchecker on each module. Audit.lean enumerates every declaration originating in these modules, including generated declarations, prints its full type, checks a body exists, rejects unsafe or partial declarations and added axioms, and permits only propext, Classical.choice and Quot.sound transitively.

Two negative controls intentionally fail by type mismatch: y membership cannot prove midpoint membership; wedge strictness cannot prove slit strictness. These tests check those specific substitutions, not general kernel soundness. REPRODUCE.md explains the portable environment variables, extraction policy and receipts. Per-process caps are 2 CPUs, 120 CPU seconds, 140 wall seconds and 6 GiB sampled private RSS, with systemd MemoryMax=6G where available. Input trees are hash-bound before and after. No campaign acceptance is inferred from successful replay.

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6. Related work, access limits and status

The target is the unqualified global assertion in the 2015 preprint [1]. Its journal version is Math. Nachr. 290 (2017), 1531-1543, DOI 10.1002/mana.201600117. The actual journal body was not obtained: the already denied Wiley route was not retried, and a repository record did not supply an accessible body. We do not assert that numbering or wording is unchanged in the journal version.

Klén [2, Sections 2, 7 and 11] explicitly discusses the boundary-point convention, gives growth bounds for the Cassinian distance near the center and compares monotonicity properties of other metrics. Those local derivative and growth bounds do not imply global Cassinian ball starlikeness. The comb example there concerns the distance-to-boundary function, rather than an explicit Cassinian ball counterexample.

The foundational 2009 paper [3] was located bibliographically through primary references and the author's CV, but no lawful primary body was obtained in this bounded check. The later coauthored paper [4] was inspected in its arXiv body: it concerns metric comparisons, Möbius distortion and the inner Cassinian metric. The scale-invariant and Möbius-invariant Cassinian metrics are different metrics; their existence and abstracts cannot resolve the assertion for (1). Their original bodies remain an access limitation here. Hariri et al. [5] compare the ordinary Cassinian and triangular ratio metrics in the unit ball. Their triangular-ratio results must not be substituted for Cassinian ball starlikeness. Similarly, the universal triangular ratio ball result in [6] is about the sum denominator, not the product in (1).

The 2020 monograph [7] was checked through its public book and chapter pages (Chapters 11 and 13), but relevant chapter bodies remained subscription content. The OpenAlex citing-work listing returned nine records for the source DOI. Their accessible arXiv bodies were screened where present. In particular, the 2025 preprint [8] studies average scale-invariant Cassinian metrics and local convexity in punctured spaces. The 2021 paper [9] and, especially, Cao-Zhang [10] remain body-unread: [10]'s abstract expressly mentions starlikeness, so this is a material unresolved prior-art gap, not clearance from an abstract. The full dated reading and access ledger is PRIOR-ART.json/md. No explicit prior counterexample or full resolution was found in the inspected bodies; that bounded negative result permits drafting but does not establish novelty.

Public proof-registry heads were checked on 1 November 2026. The retained prior review and new receipts distinguish head observations from actual body coverage; registry metadata alone is not novelty evidence. Credit for this lane remains zero. The campaign count remains 12/20 beyond the unchanged historical 39. Root alone may accept a result after resolving remaining literature and registry qualifications.

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