An explicit non-collapsing facet of the 4 × 4 × 5 marginal cone

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Abstract. We give an integral supporting functional for the no-three-way-interaction marginal cone P^{4,4,5}. Its zero face has dimension 43 in a cone of dimension 44. Five distinct labelled layer zero patterns obstruct every state-map pullback to a 4 × 4 × 4 table. Thus the collapsing conjecture of Eriksson, Fienberg, Rinaldo and Sullivant [1, Conjecture 14] fails. An explicit determinant-one minor supplies a finite ordinary facet proof. A byte-preserved Lean certificate proves the negation of the universal conjecture, with a reproducible strict build and declaration audit.

The tropical explanation is logically separate: the five layers describe maximal cells of a regular subdivision of $\Delta_3 \times \Delta_3$, or equivalently pseudovertices of the associated tropical complex. We derive the marginal-cone bridge here; we do not attribute that statistical statement to the tropical literature. The bounded prior-art review located a relevant reported enumeration by Schröter, but no matching coordinate representative. No priority claim is made.

1. The model and its cone

For positive integers p,q,r, write $[d]=\{0,\dots,d-1\}$. The hierarchical model $\Delta=[12][13][23]$ has no three-factor effect: its strictly positive distributions have log probabilities of the form $\alpha_{ij}+\beta_{ik}+\gamma_{jk}$ (including a normalization constant in these parameters). The sufficient statistic A_Δ sends a real table x to $m=(m^{12},m^{13},m^{23})$, with $m^{12}_{ij}=\Sigma_k x_{ijk}$, $m^{13}_{ik}=\Sigma_j x_{ijk}$ and $m^{23}_{jk}=\Sigma_i x_{ijk}$.

The marginal cone is $P^{p,q,r}=\{A_\Delta x: x\geq 0\}$ in $V=R^{pq+pr+qr}$. Its columns are $v_{ijk}=e^{12}_{ij}+e^{13}_{ik}+e^{23}_{jk}$, so P is their nonnegative hull. All dimensions below mean dimensions of linear spans. A supported face is $P\cap\ker \ell$ with $\ell\geq 0$ on P; a facet has dimension $\dim P-1$. Ambient codimension in redundant margin coordinates is not the facet criterion.

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2. Collapsing and the exact conjecture

An elementary collapsing of one variable merges a nonempty subset S of its d states to a single state, retaining the other states, so its new size is d−|S|+1. State relabellings are harmless and allowed. A collapsing is a finite composition of elementary collapsings. It is represented by three composite state maps s_1,s_2,s_3. On margins the induced linear map π sums entries over the corresponding fibers; for example $(\pi m)^{12}{}_{i'j'}=\Sigma_{\{s_1(i)=i',s_2(j)=j'\}} m^{12}{}_{ij}$, and similarly for the other two margins.

A facet F of the source cone is obtained by collapsing from a target facet F′ if $F=\{m\in P:\pi m\in F'\}$. The intersection with the source cone is part of this definition. In particular, a preimage in all of V is not intended. Direct substitution gives $\pi(v_{ijk})=v'_{\{s_1(i),s_2(j),s_3(k)\}}$. Fiber sums compose because summing over composite fibers is an iterated finite sum.

Conjecture 14 of the journal [1], corresponding to conj:col in the arXiv source, states: for all positive integers p≤q≤r, every facet of P^{p,q,r} is obtained by collapsing from a facet of P^{p,q,q}. Equivalently, ∀p,q,r>0 with p≤q≤r, ∀F, Facet(P^{p,q,r},F) implies the existence of a collapsing π to (p,q,q) and a facet F′ of P^{p,q,q} with $F=P^{p,q,r}\cap\pi^{-1}(F')$. These are the prose and quantified versions used here.

The MathDB record [2] states the same target dimensions, calling P a “polytope.” We use the cone explicitly defined in [1, Section 4]. The earlier assertion that all facets come from binary tables is different: [1, Example 13] already supplies an obstruction at 4 × 4 × 4 to that stronger assertion. A facet at 4 × 4 × 4 alone cannot refute Conjecture 14, since its target is itself.

Theorem. The functional in Section 3 exposes a facet F of P^{4,4,5} which is not a pullback of any subset of the target margin space under any three state maps [4]→[4], [4]→[4], [5]→[4]. Consequently it is not obtained by collapsing from a facet of P^{4,4,4}, and Conjecture 14 is false.

The stronger arbitrary-map, arbitrary-target-set assertion is an ordinary consequence of the same pigeonhole proof. The literal Lean endpoint formalizes the exact conjecture and the required facet obstruction; it does not separately package the stronger assertion as an endpoint.

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3. The explicit functional and validity

All indices start at zero. Define $\ell(m)=\Sigma_{ij}a_{ij}m^{12}{}_{ij}+\Sigma_{ik}b_{ik}m^{13}{}_{ik}+\Sigma_{jk}c_{jk}m^{23}{}_{jk}$ by the following matrices (rows in their displayed order).

```
a = [0 0 0 0]     b = [ 0  0  0  0 0]     c = [1 1 1 1 0]
    [0 0 0 1]         [-1  0  0  0 0]         [1 1 1 0 0]
    [0 0 1 1]         [-1 -1 -1  0 0]         [1 0 0 0 0]
    [0 1 1 0]         [ 0 -1  0 -1 0]         [0 1 0 1 0]
```

Put $w_{ijk}=\ell(v_{ijk})=a_{ij}+b_{ik}+c_{jk}$. Its five 4 × 4 slices, with rows i and columns j, are:

```
k=0         k=1         k=2         k=3         k=4
1 1 1 0     1 1 0 1     1 1 0 0     1 0 0 1     0 0 0 0
0 0 0 0     1 1 0 2     1 1 0 1     1 0 0 2     0 0 0 1
0 0 1 0     0 0 0 1     0 0 0 0     1 0 1 2     0 0 1 1
1 2 2 0     0 1 0 0     1 2 1 0     0 0 0 0     0 1 1 0
```

Every entry is 0, 1 or 2. Thus $\ell(A_\Delta x)=\Sigma_{ijk}x_{ijk}w_{ijk}\geq 0$ for $x\geq 0$. Let $F=P^{\{4,4,5\}}\cap\ker\ell$. It is a face. Moreover, for any nonnegative representation of a point of F, every positive coefficient must occur at a zero of w: a sum of nonnegative terms is zero only when each term is zero. Hence F is precisely the nonnegative hull of the 45 tight columns. In particular $v_{ijk}\in F$ if and only if $w_{ijk}=0$.

Write $Z_k=\{(i,j):w_{ijk}=0\}$. The sizes are 9,8,8,9,11. Equal sizes do not imply equal labelled sets: the displayed slices show $Z_0\neq Z_3$ and $Z_1\neq Z_2$ as well. Section 5 lists a distinguishing position for every pair, so the non-collapsing argument needs no classification of state maps.

The following dimension argument is over R and uses exact integer data. It is independent of any floating-point hull, tropical correspondence, or enumeration of candidate height matrices.

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4. A complete facet certificate

The 13 consistency forms are $R_i=\Sigma_j m^{12}_{ij}-\Sigma_k m^{13}_{ik}$ (four forms), $S_j=\Sigma_i m^{12}_{ij}-\Sigma_k m^{23}_{jk}$ (four), and $T_k=\Sigma_i m^{13}_{ik}-\Sigma_j m^{23}_{jk}$ (five). They vanish on every column. A linear dependence with coefficients u_i,v_j,t_k has coordinate coefficients u_i+v_j, $-u_i+t_k$, $-v_j-t_k$. Vanishing forces all $u_i=t_k=u$ and all $v_j=-u$. Thus there is exactly one dependence and the forms have rank 12. Consequently dim $P\le 56-12=44$.

Take the following 43 tight cells, in the stated order. Let B be their 56×43 column matrix, ordering coordinates first by 12, then 13, then 23, row-major in each block.

```
(0, 0, 4), (0, 1, 3), (0, 1, 4), (0, 2, 1), (0, 2, 2)
(0, 2, 3), (0, 2, 4), (0, 3, 0), (0, 3, 2), (0, 3, 4)
(1, 0, 0), (1, 0, 4), (1, 1, 0), (1, 1, 3), (1, 1, 4)
(1, 2, 0), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 0)
(2, 0, 0), (2, 0, 1), (2, 0, 2), (2, 0, 4), (2, 1, 0)
(2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 2, 1), (2, 2, 2)
(2, 3, 0), (2, 3, 2), (3, 0, 1), (3, 0, 3), (3, 0, 4)
(3, 1, 3), (3, 2, 1), (3, 2, 3), (3, 3, 0), (3, 3, 1)
(3, 3, 2), (3, 3, 3), (3, 3, 4)
```

The ordered coordinate rows below select a square submatrix N of B. Its determinant is 1. To specify a directly checkable arithmetic certificate, start $N^{(0)}=N$ and $d_{-1}=1$. At step $k=0,\dots,41$, let $d_k=N^{(k)}_{kk}$; replace each entry $i,j>k$ by $(d_k N^{(k)}_{ij}-N^{(k)}_{ik}N^{(k)}_{kj})/d_{k-1}$, and zero entries $i>k,j=k$. No swaps occur. This is fraction-free Gaussian elimination; its final diagonal entry is det N. The row list and every pivot are:

```
rows: 0 1 19 17 18 2 44 16 3 48 4 21 5 24 36 25
22 23 41 7 8 26 27 28 30 37 38 29 10 42 11 43
32 34 35 13 14 39 15 40 33 47 46
pivots:
1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 1 1 1 1 1
1 1 1 1 1 -1 1 -1 -1 1 -1 -1 -1 -1 -1 1
1 1 1 1 1 -1 -1 1 1 1 1
```

This finite recurrence and the column definition verify the determinant claim without an unspecified rank oracle; tools/certificate.py reproduces it. Therefore the 43 tight columns are independent. Since $\ell(v_{000})=1$, adjoining v_{000} gives 44 independent columns. Thus dim $P=44$, ℓ is nonzero on span P, and $\dim(\text{span } P\cap\ker \ell)=43$. The tight columns give dim $F\ge 43$, while $F\subseteq \text{span } P\cap\ker \ell$ gives dim $F\le 43$. Hence F is a facet. This completes the facet proof.

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5. The obstruction to collapsing

For completeness, these positions distinguish all ten pairs of layers. Each line gives (k,l), then a position (i,j) where exactly one of the two evaluations vanishes. They are read directly from Section 3.

```
(0,1): (0,2)    (0,2): (0,2)    (0,3): (0,1)
(0,4): (0,0)    (1,2): (0,3)    (1,3): (0,1)
(1,4): (0,0)    (2,3): (0,1)    (2,4): (0,0)
(3,4): (0,0)
```

Assume $F=\{m\in P:\pi m\in E\}$ for some subset E of the target margin space and maps $s_1:[4]\to[4]$, $s_2:[4]\to[4]$, $s_3:[5]\to[4]$. By the pigeonhole principle there are $k\neq l$ with $s_3(k)=s_3(l)$. For every i,j, the column-image formula gives $\pi(v_{ijk})=\pi(v_{ijl})$. Both original columns belong to P. Therefore $v_{ijk}\in F$ if and only if $v_{ijl}\in F$. By validity and the definition of F, this says $w_{ijk}=0$ if and only if $w_{ijl}=0$ for all i,j, contradicting the displayed distinguishing position.

This proof uses neither surjectivity of the maps, nor elementary-merge structure, nor any property of E. It therefore excludes every state map to these target sizes, and in particular every source-defined collapsing and target facet. If variable permutations were also allowed, the target has four states in every variable, so the original five-state factor would still identify two layers and give the same contradiction. No general coordinate-mixing cone automorphism is part of the source definition.

Specializing the universal conjecture to p=q=4,r=5 would assert that this very facet is obtained from P^{4,4,4}. The contradiction refutes the universal statement. This is not an assertion that facets never stabilize: for fixed p,q only finitely many labelled zero patterns can occur. The proposed stabilization threshold q is what fails. Rinaldo [3, Section 4.3.1] already explains the finite-pattern argument following Conjecture 4.3.5.

The result makes no MLE claim for a specific observed table. The relevance to likelihood is the marginal-cone facial description in [1]; the counterexample itself is a statement about facets of that cone. Neither semigroup normality nor Markov-basis degree bounds are substituted for this facial statement.

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6. Lean correspondence and trust boundary

The dossier preserves proof/MathDB378395.lean byte for byte from the frozen author packet. The entire mathematical development is in that module; ReviewAudit.lean and ReviewSanity.lean are labelled reviewer supplements. The module hash is b047f65580922c920bb63149632bb79ad11378c394f72da870b7da315482b392. Names below are in MathDB378395, with Cert or Main as indicated.

Ordinary object / claim	Lean definition / theorem
Cell, redundant margins	Cell, Margin
A_Delta and positive hull	margins, marginalCone
supporting face, facet	IsFace, IsFacet
state fiber-sum map	collapseMap
elementary merges / sequences	IsElementaryMerge, IsCollapsing
facet pullback	ObtainedByCollapsing
universal conjecture	CollapsingConjecture
column-image formula	collapseMap_col
functional / zero face	Cert.ellZ; Main.ell, Main.F
integer validity	Cert.ell_nonneg
43 tight independent columns	Cert.bcell, phiZ, phi_biorth
12 independent relations	Cert.psiZ, psi_cols, psi_piv
cone / face dimensions	Main.finrank_span_cone / _F
facet / obstruction	Main.F_isFacet / F_not_collapsed

The literal endpoint is MathDB378395.Main.collapsing_conjecture_false : ¬MathDB378395.CollapsingConjecture. The witness endpoint exists_noncollapsible_facet existentially supplies F, its facet proof and its non-collapsing proof. CollapsingConjecture quantifies over natural numbers with $0<p\le q\le r$, and over every facet; the target is exactly (p,q,q). The theorem supplies no unproved geometric bridge or external rank hypothesis.

Lean verifies the integer tables by decide +kernel. It transports the identities to R and uses biorthogonal integer functionals to prove the 43-column independence. The ordinary determinant proof in Section 4 is an alternative finite certificate, not an additional Lean theorem. Rank-nullity and twelve margin forms bound the cone; adding ℓ bounds the face. The collapsed_slices_agree lemma is the pigeonhole implication used at the endpoint.

Replay pins Lean 4.34.1, commit 5045d0056413266e57c625dcd7c365b10e377c52, and mathlib d13f23b723b8a846827a245b89c10fc7d3f11612. The audit covers all 134 module declarations, including generated constructors and recursors, requiring bodies where appropriate and rejecting unsafe/partial declarations. Only propext, Classical.choice and Quot.sound are allowed transitively. Cached upstream dependency oleans are trusted; this is not an upstream rebuild or a source-to-cache attestation.

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7. Tropical interpretation: an explanatory bridge

Let h=a be heights on the vertices (e_i,e_j) of $\Delta_3 \times \Delta_3$. Adopt the lower-hull convention. Consider $Q_h=\{(y,z):y_i+z_j\le h_{ij}\}$, modulo $(y,z)\mapsto(y+t1,z-t1)$. A supporting potential has tight set $\{(i,j):y_i+z_j=h_{ij}\}$. Develin–Sturmfels [4, Lemma 22 and the following paragraph] identify the face/subdivision duality; vertices of Q_h correspond to full-dimensional maximal subdivision cells. The bounded-face complex gives the tropical complex. Its vertices are pseudovertices, not necessarily the original tropical generators.

In our notation $y_i=-b_{ik}$ and $z_j=-c_{jk}$. Thus every slice gives a supporting potential and its tight set is exactly Z_k. The independent exact reviewer enumeration in candidate-subdivision.json checks all 11,440 seven-edge subsets of $K_{4,4}$: connected ones determine spanning-tree potentials up to gauge, and the sixteen inequalities are tested. It finds exactly the five maximal cells $Z_0,\dots,Z_4$, of sizes 9,8,8,9,11. These diagnostics explain the geometry; the facet proof does not depend on their exhaustiveness.

Here is the additional marginal-cone deduction. For general p,q and m maximal full-dimensional cells, include one potential pair $(-b_k,-c_k)$ per cell. The dual marginal cone is $\{(h,b,c):h_{ij}+b_{ik}+c_{jk}\ge 0\}$, modulo the evaluation kernel L. Solving all tight equations projects onto heights affine on each cell. Strict inequalities at the given h identify this height space with the linear span of its secondary cone. Each tight graph is connected and spanning, so each layer potential has exactly one scalar gauge. If the secondary cone has dimension d modulo affine heights, the tight-equation solution space has dimension $(p+q-1)+m+d$. The evaluation kernel has dimension $p+q+m-1$: its elements are precisely coefficients making all column evaluations zero, as the same coordinate comparison used in Section 4 shows. The quotient tight face therefore has dimension d.

For a nontrivial coarsest regular subdivision, d=1. Its m represented cells consequently give an extreme dual ray, hence a marginal-cone facet. This is a sufficient construction, not a bijection for all facets; coordinate margins and repeated layers need separate treatment. Herrmann–Joswig–Speyer [7, Section 3] relate rigidity to secondary rays. Their Stiefel lift (Theorem 7, Corollary 14) and the vertex-figure machinery of [6, Section 3] concern hypersimplices: hypersimplex cell counts cannot replace product-cell counts. Splits in [5] have two cells and alone do not provide this example. None of these papers is cited as containing the statistical deduction just derived.

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8. Diagnostics, related work and limitations

The author reports a normalized {0,1} height scan finding 54 five-cell and 6 seven-cell examples, verified as facets at the corresponding table sizes. These are labelled height matrices, not asserted symmetry-orbit counts. Smaller scans were consistent with [1, Table 1], which stops at $4 \times 4 \times 4$ and does not test $4 \times 4 \times 5$. These reports are discovery diagnostics only; no exhaustive classification or new formal theorem about the scan is claimed.

Schröter [8, Section 4, Propositions 37–38, Tables 1–2] and [9, Section 3.4] report computing all coarsest regular subdivisions of the relevant small products, including $\Delta_3 \times \Delta_3$. The 39 classes at d=4,n=8 count symmetry classes after Stiefel lifting into the hypersimplex, not product-subdivision orbits. No matching representatives or maximal-cell statistics were located in the inspected bodies. The reported completeness conditionally includes this subdivision, but no explicit orbit identification was made. This is the closest computational prior art and a substantial limitation on novelty claims.

The independent review held novelty/significance; root subsequently issued a bounded qualified pass for manuscript preparation, with zero credit. Both assessments are preserved verbatim. A comparison fingerprint (positive scaling, affine row/column shifts and global symmetries) is supplied in the dossier; it is not proof of an older match. No priority is asserted. The source's open label is not novelty evidence.

Rinaldo [3] restates the exact conjecture. Fienberg–Rinaldo [10, Example 3] uses the $4 \times 4 \times 4$ facet count and discusses remaining facial-classification questions; it supplies no $4 \times 4 \times 5$ refutation in the reviewed passages. The frozen root adjudication additionally reports that Dobra et al., "Algebraic statistics and contingency table problems: log-linear models, likelihood estimation, and disclosure limitation" (2009), flag the conjecture as open. That report is disclosed as adjudication-supplied evidence: this dossier has no independently read Dobra primary-body copy. These later statements do not settle priority.

References (primary texts and dated retrieval receipts are retained in the dossier).

[1] N. Eriksson, S. E. Fienberg, A. Rinaldo, S. Sullivant. Polyhedral conditions for the nonexistence of the MLE for hierarchical log-linear models. J. Symbolic Comput. 41 (2006), 222–233. arXiv:math/0405044, Section 4. [2] MathDB 378395, frozen JSONL record; source status as of May 2004. [3] A. Rinaldo. On Maximum Likelihood Estimation for Contingency Tables. PhD thesis, Carnegie Mellon University, 2006, Section 4.3.1.

[4] M. Develin, B. Sturmfels. Tropical convexity. Doc. Math. 9 (2004), 1–27; arXiv:math/0308254. [5] S. Herrmann, M. Joswig. Splitting polytopes. Münster J. Math. 1 (2008), 109–142; arXiv:0805.0774. [6] S. Herrmann, A. Jensen, M. Joswig, B. Sturmfels. How to draw tropical planes. Electron. J. Combin. 16 (2009), R6; arXiv:0808.2383.

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