# A 34-vertex maximal rigid diameter-5 subset of the eight-dimensional Hamming cube

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## Abstract

We give an explicit set of 34 vertices in the eight-dimensional Hamming cube that has diameter exactly 5, is inclusion-maximal subject to diameter at most 5, and has a diameter partner at every vertex. Hence $b'(8;5) \geq 34 > 2^5$, contradicting Conjecture 1 of Briggs, Feng and Wells [1], in its stated universal form. The finite verification is supplied as an ordinary exhaustive algorithm and as a Lean proof over the entire cube. An explicit equivalence transfers the Boolean encoding to the source's integer-sign cube. We do not determine the exact extremal value, minimal counterexample parameters, or a general family. The related-work assessment is bounded and carries no priority claim.

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## 1. Definitions and statement

Let $Q_n = \{-1,+1\}^n$, with coordinates indexed from 0 to n-1. Its Hamming distance d(x,y) is the number of coordinates on which x and y differ. For a nonempty subset S, its diameter is max{d(x,y): x,y in S}. Following [1, Section 2], S is maximally diameter-r when its diameter equals r and adjoining any x outside S produces diameter greater than r. It is rigid when every x in S has a y in S with d(x,y)=r. Here maximal means inclusion-maximal, not largest cardinality.

Write $b'(n;r)$ for the largest cardinality of a maximally diameter-r rigid subset of $Q_n$. Section 6, Conjecture 1 of [1] states $b'(n;r)=2^r$ whenever $n \geq r$. All maxima here are over finite sets. The coordinate r-subcube with remaining coordinates fixed has size $2^r$ and supplies an admissible set: an outsider differs in a fixed coordinate, and choosing all free coordinates opposite to it gives distance greater than r. Every member has its free-coordinate antipode at distance r.

Theorem. There exists a maximally diameter-5 rigid subset of $Q_8$ of cardinality 34. Consequently the universal conjecture in [1] is false.

All parameters in the contradiction are positive and satisfy $r \leq n$. Natural-number quantifiers in Lean also cover r=0: a nonempty singleton has exact diameter zero and is rigid, whereas the empty set has no pair attaining the diameter. No edge convention about the empty set or dimension zero is used in the counterexample.

## 2. The literal witness and a compact certificate

For $0 \le k < 256$, let $v(k)_i$ be +1 if $\mathrm{floor}(k/2^i) \bmod 2 = 1$ and -1 otherwise, for $0 \le i < 8$. The least significant digit is coordinate zero. Define $A=\{v(k): k \text{ in } L\}$, where L is the following increasing list.

```
L = [0, 3, 5, 8, 10, 11, 12, 15, 18, 23, 26, 30,
     66, 68, 71, 74, 78, 79, 130, 131, 132, 135, 137,
     138, 140, 142, 143, 150, 158, 194, 198, 203, 206, 217]
```

The distinct integers in L lie between 0 and 255. Uniqueness of eight-digit binary expansion shows that their encodings are distinct, so $|A|=34$. For an integer k in this range define $e_i(k)=\mathrm{floor}(k/2^i) \bmod 2$ and $D(k,l)$=sum over i=0,...,7 of $[e_i(k) \neq e_i(l)]$, where the bracket is 1 when its condition holds and 0 otherwise. Then $D(k,l)=d(v(k),v(l))$.

The following complete algorithm is a compact verifiable description of the certificate. Every loop has explicit finite bounds. It requires only division, remainder, integer comparison, addition and maxima over 34 entries. It generates the full 256-row certificate rather than relying on a compressed or externally trusted witness list.

```
digit(k,i) = (k // (2**i)) % 2
D(k,l) = sum(digit(k,i) != digit(l,i) for i in range(8))
H = [0]*9
for k in L:
    for l in L:
        H[D(k,l)] += 1
assert H == [34,72,276,308,276,190,0,0,0]
for k in range(256):
    M = max(D(k,l) for l in L)
    partner = min(l for l in L if D(k,l) == M)
    assert M == 5 if k in L else M > 5
    output(k, M, partner)
```

The output is retained in independent-certificate.json; scripts/check_witness.py is the executable version. For example, D(0,217)=5. The ordered-pair histogram has total 1,156=34 squared, with 190 ordered pairs at distance 5 and none at greater distance. Its diagonal contributes exactly the 34 distance-zero pairs. Each member's maximum is 5; every outside maximum is 6, 7 or 8. Those row conditions, rather than the histogram alone, certify rigidity and maximality.

This computation was independently implemented by coordinate comparison without importing the author's checker. The original author-certificate.json is retained separately. Python provides a readable finite check; the formal theorem does not assume the Python output.

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## 3. Proof and interpretation of the certificate

Proof of the theorem. First, the digit map is a bijection between the integers 0,...,255 and the signed cube $Q_8$. Indeed every such integer has a unique eight-digit binary expansion, and replacing digits 0 and 1 by -1 and +1 is bijective coordinatewise. Thus the outer loop in Section 2 covers all cube vertices, including every point not in A.

The histogram computation counts each ordered pair from A exactly once. Its entries above distance 5 vanish, proving $d(x,y) \le 5$ for every x,y in A. Since the distance-5 count is positive, some pair attains 5; alternatively the explicit pair v(0),v(217) does so. Hence the diameter is exactly 5, not merely bounded above by 5.

For each k in L the second loop gives M=5 and supplies an index l in L attaining that maximum. Therefore each v(k) has a partner v(l) at distance exactly 5. This establishes rigidity with respect to the exact diameter. The existence of some diameter pair alone would not prove rigidity; the row condition is needed for all 34 members.

For each k outside L the same loop supplies l in L with D(k,l)>5. Adjoining v(k) therefore produces a set whose diameter exceeds 5. This proves the source's maximality requirement. More generally any proper superset T of A contains such an outsider, so T cannot have diameter at most 5. It is unnecessary to enumerate all supersets.

The three facts prove that A is a maximally diameter-5 rigid set. The finite collection of attainable cardinalities therefore contains 34, so its greatest element is at least 34. Since $34>32=2^5$, evaluating the asserted universal equality at (n,r)=(8,5) yields a contradiction. This proves the theorem and refutes the conjecture. The argument does not claim that 34 is the greatest attainable cardinality.

## What the finite check proves

The ordinary proof can be checked directly from the list and the bounded algorithm; the Lean proof certifies exactly the corresponding quantified predicates. The histogram and the 256-row table are explanatory diagnostics, whereas the quantified Lean lemmas are the certificate accepted by the kernel. Agreement of two programs is useful error detection but is not substituted for the formal theorem.

In Vietoris-Rips terminology, a nonempty set of pairwise distance at most 5 is a simplex in the scale-5 clique complex. The outside exclusion makes it a facet. Rigidity is the diameter-partner condition used in [1], not Euclidean rigidity or affine independence. No assertion about homology, homotopy type or affine dimension is required. The sole extremal conclusion here is the lower bound $b'(8;5) \ge 34$.

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## 4. Formal correspondence

All names below lie in namespace MathDB364205. The dossier preserves Cube.lean and Signed.lean byte for byte from the author packet. Reviewer diagnostics are separately labelled. The Boolean and signed formulations are related by an explicit equivalence, not by a convention left outside the formal proof.

| Lean declaration(s) | Mathematical content |
|---|---|
| Cube; dist | Boolean coordinate cube and cardinality of the mismatch-coordinate set. |
| MaximalDiameter; Rigid | Pairwise bound, an attained diameter, outside exclusion; diameter partner for every member. |
| vertex; A; card_A | Eight binary digits; the displayed finite set; exactly 34 distinct vertices. |
| pair_bound; rigid_A | All pairwise distances at most 5; every member attains distance 5. |
| outside_A; diameter_attained; maximal_A | Every cube outsider has a member at distance greater than 5; a pair at 5; assembled maximality. |
| SourceClaim; source_refutation | For all natural n,r with r ≤ n, 2^r is greatest among admissible Boolean cardinalities; its negation. |
| Sign; SignedCube; signEquiv | Integer subtype z=-1 or z=1; signed coordinate functions; Boolean/sign equivalence. |
| cubeEquiv; signedDist; dist_encoding | Coordinatewise equivalence; signed mismatch count; exact equality of distances. |
| signedA; signedA_card | Image of A under the equivalence; cardinality preserved by injectivity. |
| SignedMaximalDiameter; SignedRigid | The same exact-diameter, outside and partner conditions on signed vertices. |
| signedA_maximal; signedA_rigid | Transport of maximality through the inverse equivalence, including arbitrary outsiders; transport of partners. |
| LiteralSourceClaim; literal_source_refutation | Universal signed greatest-cardinality assertion; its literal negation, using 34 ≤ 2^5 as the contradiction. |

The greatest-element formulation contains both attainment and an upper bound on every attainable size. The final proof uses the upper-bound clause at (8,5), with signedA as the attaining set of size 34. Finite cubes identify this statement with the largest-size formulation of [1]. Kernel decide proves the finite Boolean lemmas by reduction; no native_decide, added axiom, or unproved Boolean-to-sign bridge is used.

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## 5. Related constructions and limits of subsumption

The independent source review supplied with the dossier gives qualified prior-art and significance assessments; this draft does not convert them into priority certification. Both versions of [1] inspected there retain Section 6, Conjecture 1. The source's Theorem 4 constructs rigid facets from Hadamard simplices, Proposition 5 takes products, and Corollary 6 adds fixed coordinates. A Hadamard simplex of diameter s has 2s vertices. For positive integral s, $2s \le 2^s$; an s-cube has exactly $2^s$ vertices. Products add diameters and multiply sizes, so products of these particular factors cannot exceed $2^r$. This is a bound on those construction mechanisms, not a classification of all rigid facets.

Ordinary diameter-bounded anticodes can be larger. For diameter 5 in $Q_8$, the canonical double ball $Q_1$ times $B_2(Q_7)$ has 2(1+7+21)=58 vertices, where $B_2$ is the ball of radius 2 about zero in Boolean coordinates. A point whose seven-coordinate part is zero has eccentricity only 1+2=3. Thus this anticode is not rigid. Keeping just the seven-coordinate weight-2 shell produces 42 points with diameter partners, but the omitted centre can still be adjoined within diameter 5; the shell is not maximal. Neither familiar operation supplies the simultaneous properties required here.

Kleitman's diameter bound and the Ahlswede-Katona extremal classification concern largest-cardinality anticodes. Bezrukov [2, Section 1 and Theorems 1-2] describes these maximum objects; despite the English title's word 'maximal', its d-maximal means maximum cardinality. At diameter 5 in $Q_8$ the ball/double-ball extremizers have the nonrigidity just noted. Their maximum-cardinality theorem therefore does not determine b'. The original Kleitman and Ahlswede-Katona papers were not independently audited in this lane; their role here is through the primary successor body [2] and the supplied review.

The 2026 anticode-counting work of Horlemann and Ravi [3] distinguishes diameter-bounded, maximum and inclusion-maximal sets. Its intersection description determines an inclusion-maximal diameter-d anticode by at most $2^{d+1}$ centres; the underlying Helly-type bounds of Alon, Jin and Sudakov [4] likewise constrain intersections of Hamming balls. For d=5 this permits 64 determining centres. It neither constructs a rigid set of size greater than 32 nor forces each member to have a diameter partner. Counting descriptions and estimating the total number of anticodes do not resolve the extremal rigid-size question.

Linear or affine binary anticodes of diameter d have dimension at most d: in reduced row-echelon form the sum of all basis rows has a 1 in every pivot position, so its weight is at least the dimension. Translation to a linear code preserves diameter. Thus a direct linear or affine diameter-5 construction has at most 32 words. The projective-linear anticode bounds and complementary constructions in Chen and Xie [5] concern that restricted setting, not arbitrary nonlinear rigid inclusion-maximal sets. Selected 2026 hypercube VR successor bodies are compared in the source review, but no exhaustive literature clearance is asserted.

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## 6. Reproducibility, audit scope and bounded claims

The independent technical audit in TECHNICAL-REVIEW.md precedes this manuscript. It rebuilds the safely extracted author sources with Lean 4.34.1, commit 5045d0056413266e57c625dcd7c365b10e377c52, and mathlib d13f23b723b8a846827a245b89c10fc7d3f11612. Strict compilation uses lean -j1 -DwarningAsError=true. Separate leanchecker runs cover each positive module. The audit enumerates all declarations by Cube/Signed module origin, including generated auxiliaries, records full types and actual body presence, checks unsafe/partial flags, and restricts transitive axioms to propext, Classical.choice and Quot.sound.

Labelled reviewer examples test small signed distances, zero-dimensional singletons, attained diameter and inclusion-maximality. Two negative controls test rejection of a false cardinality and a false signed distance. These tests do not establish general soundness of the compiler or eliminate the trusted external dependency boundary. Prebuilt pinned upstream oleans and the compiler are trusted; this dossier contains no full upstream rebuild or independent source-to-olean attestation.

REPRODUCE.md documents a portable driver using LEAN_BIN and LEAN_PACKAGES. It refuses existing oleans, applies bounded CPU, wall and memory controls, records receipts and removes only its own generated files. A fresh extraction of the delivered source-only ZIP is replayed separately. The full dossier retains raw MathDB bytes and their hash, primary-source copies and retrieval receipts, frozen input bindings, proof modules, independent audit logs and certificate diagnostics. No external service is required to rerun the Lean certificate with the pinned cache.

The result establishes a single explicit obstruction to the proposed universal bound. It does not determine b'(8;5), a minimum counterexample, classification, asymptotics or an infinite family. No publication, peer review, worldwide novelty or priority is claimed. The bounded source review reports qualified passes for prior art, novelty and significance. Campaign accounting remains 7/20 beyond 39 unchanged; this draft carries zero accepted credit, and root alone accepts.

## References


[1] J. Briggs, Z. Feng and C. Wells, *Facets in the Vietoris-Rips complexes of hypercubes*, arXiv:2408.01288v1/v2 (2024), Sections 2, 3 and 6; Conjecture 1. Frozen primary HTML and dated receipts accompany this draft.

[2] S. L. Bezrukov, *Description of Maximal Subsets of a Given Diameter in a Hamming Space*, Problems of Information Transmission 23:1 (1987), 106-109 in the Russian source. Math-Net record ppi768; full four-page primary body retained. Extremal classification compared through this body.

[3] A.-L. Horlemann and A. Ravi, *The Asymptotic Number of Anticodes in Hamming Spaces*, arXiv:2610.31618v1 (2026), Definitions 2.1/2.4, Lemmas 2.6, 3.1/3.2 and Theorem 3.11. Primary body retained under sources/r1-anticode-count.body.

[4] N. Alon, Z. Jin and B. Sudakov, *The Helly number of Hamming balls and related problems*, arXiv:2405.10275v2 (2024), Theorems 1.1-1.2. Primary body and receipt retained.

[5] H. Chen and C. Xie, *Linear Codes from Projective Linear Anticodes Revisited*, arXiv:2406.07112v1 (2024), Theorem 2.1, Theorem 3.1 and Corollary 3.1. Primary body retained under sources/r1-linear-anticode.body.


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