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A six-point ordinal space with the majorization property that does not embed in the real line

A Lean-certified counterexample to Conjecture 7.4 of Keller and Petrov (MathDB 366371)

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Abstract. Keller and Petrov introduced ordinal spaces, which record only whether one pair of points is closer than, as close as, or farther than another. They showed that an ordinal space embeddable in the real line has an enumeration with a *majorization property*, proved the converse for at most four points, and conjectured the converse in general (Conjecture 7.4 of [1]). We give a six-point ordinal space, defined by a strict ranking of its 15 pairs, whose identity enumeration has the majorization property for chains of every length, with repeated indices and arbitrary permutations, but which does not embed in the line. Majorization for all chain lengths is reduced, by counting pieces above each threshold, to a finite check over 63 × 63 pairs of strictly increasing lists. Non-embeddability follows from five comparisons whose sum is a strict inequality between two equal linear forms. The negation of the conjecture, with literal definitions, is proved in Lean 4 with Mathlib using only the standard axioms. The phenomenon is the analogue, for ordinal spaces on the line, of classical results in measurement theory; we make no priority claim beyond a bounded search.

1. Introduction

Ordinal data analysis works with comparisons of dissimilarities rather than their values. Keller and Petrov [1] axiomatize this setting: an *ordinal space* is a set X with a map δ that, for every two pairs {x, y} and {z, w}, says which pair is closer. A natural question is when such a space is the ordinal shadow of points on the real line. For this, [1, Section 7] defines a combinatorial condition on an enumeration $x_1, \ldots, x_n$ of the points, the majorization property (Definition 2.3 below). Proposition 7.3 of [1] shows that every space embeddable in $\mathbb{R}^1$ has such an enumeration, and Theorem 7.5 shows the converse when |X| = 4. The authors write: “We are not able to say whether the existence of an enumeration with majorization property is also sufficient for embeddability in $\mathbb{R}^1$ if n > 5, but for n ≤ 4 we have sufficiency”, and state the converse as Conjecture 7.4. MathDB records this as problem 366371 [7].

Main result. Conjecture 7.4 is false, already for six points (Theorem 3.1). The witness is a strict ranking of the 15 pairs of {0, …, 5}. Two features make the proof short and fully checkable. First, the majorization property quantifies over chains of every length k, but for this space it reduces exactly to a finite statement about strictly increasing lists and thresholds (Section 4). Second, once the order of the points on a putative line is forced, five comparisons add up to the contradiction 0 < 0 (Section 5). Both parts, and the negation of the conjecture with the source definitions, are kernel-checked in Lean (Section 6).

Context. Finite representation problems of this kind are classical in measurement theory, where representability is characterized by cancellation conditions, that is, by solvability of linear inequalities [2, 3], and where Kraft, Pratt and Seidenberg [4] gave a five-atom qualitative probability that satisfies de Finetti’s axioms but has no agreeing measure. The present result is of that type, and is expected in spirit. It is not, however, a consequence of those results (Section 7). It settles the case left open in [1] with a small explicit witness; its depth is modest.

2. Definitions and the conjecture

We follow [1] (arXiv v1, which we read; see Section 8). Write δ(x, y) < δ(z, w) for δ(x, y, z, w) = ‘<’, and similarly for = and >.

Definition 2.1 ([1, Definition 1.1]). Let X be a nonempty set and δ : X × X × X × X → {‘<’, ‘=’, ‘>’}. Then (X, δ) is an *ordinal space* if for all x, y, u, v, z, w ∈ X:

(i) $\delta(x, y, x, y) = \text{‘=’}$;

(ii) $\delta(x, y, z, w) = \delta(y, x, z, w) = \delta(x, y, w, z)$;

(iii) $\delta(x, y, z, w) = -\delta(z, w, x, y)$, where $-\text{‘<’} = \text{‘>’}$, $-\text{‘>’} = \text{‘<’}$ and $-\text{‘=’} = \text{‘=’}$;

(iv) $\delta(x, y, u, v) = \delta(u, v, z, w) = \text{‘=’}$ implies $\delta(x, y, z, w) = \text{‘=’}$;

(v) $\delta(x, y, u, v) = \text{‘<’}$ and $\delta(u, v, z, w) \in \{\text{‘<’}, \text{‘=’}\}$ imply $\delta(x, y, z, w) = \text{‘<’}$;

(vi) $\delta(x, y, u, v) \in \{\text{‘<’}, \text{‘=’}\}$ and $\delta(u, v, z, w) = \text{‘<’}$ imply $\delta(x, y, z, w) = \text{‘<’}$;

(vii) $\delta(x, x, z, w) = \text{‘<’}$ if $z \neq w$, and $\delta(x, x, z, w) = \text{‘=’}$ if $z = w$.

Definition 2.2 ([1, (7.1)–(7.3)]). (X, δ) *embeds in the line*, written $X \hookrightarrow \mathbb{R}^1$, if there is $\Phi : X \to \mathbb{R}$ such that for all $x, y, z, w \in X$, $\delta(x, y) <, =, > \delta(z, w)$ holds if and only if $|\Phi(x) - \Phi(y)| <, =, > |\Phi(z) - \Phi(w)|$, respectively.

For $n \in \mathbb{N}^+$ let $\mathscr{I}_n$ be the set of tuples $(i_0, i_1, \ldots, i_k) \in \{1, \ldots, n\}^{k+1}$ with $k \in \mathbb{N}^+$ and $i_0 \le i_1 \le \cdots \le i_k$; repeated indices are allowed. We call such a tuple a *chain* of length k.

Definition 2.3 ([1, Definition 7.1]). Let $x_1, \ldots, x_n$ be an enumeration of X, $|X| = n$, and let $(i_0, \ldots, i_k), (j_0, \ldots, j_k) \in \mathscr{I}_n$ have the same length k. Write (i) ~ (j), respectively (i) ≺ (j), if there is a permutation π of $\{1, \ldots, k\}$ such that

(7.4) $\delta(x_{i_{l-1}}, x_{i_l}) = \delta(x_{j_{\pi(l)-1}}, x_{j_{\pi(l)}})$ for all $l \in \{1, \ldots, k\}$, respectively

$\delta(x_{i_{l-1}}, x_{i_l}) \le \delta(x_{j_{\pi(l)-1}}, x_{j_{\pi(l)}})$ for all $l \in \{1, \ldots, k\}$, but not (7.4).

The enumeration has the *majorization property* if for every $k \in \mathbb{N}^+$ and all chains (i), (j) of length k, (i) ≺ (j) implies $\delta(x_{i_0}, x_{i_k}) < \delta(x_{j_0}, x_{j_k})$, and (i) ~ (j) implies $\delta(x_{i_0}, x_{i_k}) = \delta(x_{j_0}, x_{j_k})$.

Conjecture 2.4 ([1, Conjecture 7.4]). *Let (X, δ) be an ordinal space with $|X| = n \in \mathbb{N}^+$ and let there exist an enumeration $x_1, x_2, \ldots, x_n$ of the points of the space X satisfying the majorization property. Then $X \hookrightarrow \mathbb{R}^1$.*

Theorem 7.5 of [1] proves that for $|X| = 4$, $X \hookrightarrow \mathbb{R}^1$ if and only if some enumeration has the majorization property (and gives a third, explicit equivalent condition); together with the easy cases this gives sufficiency for $n \le 4$. The quoted sentence leaves $n = 5$ formally unaddressed; our witness has $n = 6$, in the open range under any reading.

Remark 2.5 (reading of “but not (7.4)”). We read it, as the Lean module does, as “not all k equalities hold for this same π”. The other possible reading, “not (i) ~ (j)”, gives the same relation: δ restricted to pairs is a total preorder and may be replaced by integer ranks; if the pieces of (i) are termwise at most those of (j) under π and the two multisets of pieces coincide, the rank sums are equal, so every termwise inequality is an equality. The diagnostic of Section 6 also checks both readings literally for $k \le 4$.

3. The witness

Let $X = \{0, 1, 2, 3, 4, 5\}$. Let D be the symmetric function on $X \times X$ with $D(x, x) = 0$ and off-diagonal values given by Table 1; D ranks the 15 pairs from 1 (closest) to 15 (farthest). Define $\delta(x, y, z, w)$ to be the comparison of the natural numbers $D(x, y)$ and $D(z, w)$. We use the enumeration $x_{i+1} = i$, so from now on chains are nondecreasing tuples in $\{0, \ldots, 5\}$.

pair	**23**	**34**	**12**	**24**	**01**	**13**	**14**	**45**	**35**	**02**	**25**	**03**	**04**	**15**	**05**
D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 1. The witness: “ab” denotes the pair {a, b}. Equivalently, the rows of the Lean matrix `D` are (0, 5, 10, 12, 13, 15), (5, 0, 3, 6, 7, 14), (10, 3, 0, 1, 4, 11), (12, 6, 1, 0, 2, 9), (13, 7, 4, 2, 0, 8), (15, 14, 11, 9, 8, 0).

> ***Theorem 3.1.*** *(X, δ) is an ordinal space with $n = 6$ points; the enumeration $x_{i+1} = i$ has the majorization property; and X does not embed in $\mathbb{R}^1$. Hence Conjecture 2.4 is false.*

Proof that (X, δ) is an ordinal space. D is symmetric, $D(x, x) = 0$, and $D(x, y) \ge 1$ for $x \neq y$. Comparison of natural numbers is a total preorder, so (i)–(vi) are reflexivity, symmetry, antisymmetry and the transitivity laws of = and < on

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$\mathbb{N}$ (with symmetry of D for (ii)), and (vii) holds since $D(x, x) = 0 < D(z, w)$ exactly when $z \neq w$. ∎

As a consistency check with [1, Proposition 7.2(ii), (iii)], which any majorizing enumeration must satisfy: $D(0, 1) < D(0, 2) < D(0, 3) < D(0, 4) < D(0, 5) = 15 > D(1, 5) > D(2, 5) > D(3, 5) > D(4, 5)$ (values 5, 10, 12, 13, 15, 14, 11, 9, 8), and $\{0, 5\}$ is the unique farthest pair. The remaining two parts are proved in Sections 4 and 5.

4. Majorization for chains of every length

For a chain $I = (i_0 \le \cdots \le i_k)$ its *pieces* are $f_I(l) = D(i_{l-1}, i_l)$ for $l = 1, \ldots, k$. A piece with $i_{l-1} = i_l$ is *degenerate* and has value 0. For a threshold t let $c_I(t)$ be the number of l with $f_I(l) \ge t$. For a list $A = (a_0, \ldots, a_r)$ let $\mathrm{adj}_t(A)$ be the number of consecutive pairs (a_{s-1}, a_s) with $D(a_{s-1}, a_s) \ge t$.

Lemma 4.1 (counting). Let I, J be chains of length k and π a permutation of $\{1, \ldots, k\}$ with $f_I(l) \le f_J(\pi(l))$ for all l. Then $c_I(t) \le c_J(t)$ for every t. If moreover $f_I(l_0) < f_J(\pi(l_0))$, then $c_I(t_0) < c_J(t_0)$ for $t_0 = f_J(\pi(l_0))$. If $f_I(l) = f_J(\pi(l))$ for all l, then $c_I = c_J$.

Proof. Let $S_I(t) = \{l : f_I(l) \ge t\}$. If $l \in S_I(t)$ then $f_J(\pi(l)) \ge f_I(l) \ge t$, so π maps $S_I(t)$ injectively into $S_J(t)$. For $t = t_0$ the image misses $\pi(l_0) \in S_J(t_0)$: by injectivity its only preimage is l_0, and $f_I(l_0) < t_0$. So the inclusion is proper. The last claim follows by applying the first to π and to π^{-1}. ∎

Lemma 4.2 (reduction). For a chain I let red(I) be the list obtained by deleting consecutive repeated entries. Then red(I) is a nonempty strictly increasing list in $\{0, \ldots, 5\}$, that is, one of the 63 nonempty subsets of $\{0, \ldots, 5\}$ listed increasingly; it starts with i_0 and ends with i_k; and $c_I(t) = \mathrm{adj}_t(\mathrm{red}(I))$ for every $t \ge 1$.

Proof. Deleting an entry i_l equal to i_{l-1} removes the degenerate piece, whose value 0 is below $t \ge 1$, and changes no other piece: the piece (i_l, i_{l+1}) becomes (i_{l-1}, i_{l+1}), the same pair. The first entry is never deleted, and the last surviving entry equals i_k. A nondecreasing list without consecutive repeats is strictly increasing. ∎

Lemma 4.3 (finite check). For all nonempty strictly increasing lists $A = (a_0 < \cdots < a_r)$ and $B = (b_0 < \cdots < b_s)$ in $\{0, \ldots, 5\}$:

(a) if $\mathrm{adj}_t(A) \le \mathrm{adj}_t(B)$ for all $t \in \{1, \ldots, 15\}$, with strict inequality for some such t, then $D(a_0, a_r) < D(b_0, b_s)$;

(b) if $\mathrm{adj}_t(A) = \mathrm{adj}_t(B)$ for all $t \in \{1, \ldots, 15\}$, then $D(a_0, a_r) = D(b_0, b_s)$.

Proof. This is a finite statement about $63 \times 63 = 3969$ pairs and 15 thresholds, verified exhaustively. In Lean it is the theorem `finite_check`, evaluated by the kernel (`decide +kernel`), and it is recomputed independently by the standard-library script `diagnostic.py`. ∎

Proposition 4.4. The enumeration $x_{i+1} = i$ has the majorization property.

Proof. Let I, J be chains of length $k \ge 1$, and put $A = \mathrm{red}(I)$, $B = \mathrm{red}(J)$. Suppose $I \prec J$ via π. Then $f_I(l) \le f_J(\pi(l))$ for all l, and the inequality is strict for some l_0, since not all of them are equalities. By Lemma 4.1, $c_I \le c_J$ everywhere and $c_I(t_0) < c_J(t_0)$ with $t_0 = f_J(\pi(l_0))$. As $t_0 > f_I(l_0) \ge 0$ and all values of D are at most 15, $t_0 \in \{1, \ldots, 15\}$. By Lemma 4.2 the same relations hold between $\mathrm{adj}_t(A)$ and $\mathrm{adj}_t(B)$ for $t \in \{1, \ldots, 15\}$, and A, B have the endpoints of I, J. Lemma 4.3(a) gives $D(i_0, i_k) < D(j_0, j_k)$, i.e. $\delta(x_{i_0}, x_{i_k}) < \delta(x_{j_0}, x_{j_k})$. If $I \sim J$, Lemma 4.1 gives $c_I = c_J$, hence $\mathrm{adj}_t(A) = \mathrm{adj}_t(B)$ for $t \in \{1, \ldots, 15\}$, and Lemma 4.3(b) gives equality of the endpoint values. ∎

For example, $(2, 3, 5) \prec (0, 2, 3)$: the pieces are (1, 9) and (10, 1), and only the transposition matches them termwise; accordingly $D(2, 5) = 11 < 12 = D(0, 3)$. Literal checks of Definition 2.3, with repeated indices and all permutations, for $k \le 4$ (82,957 pairs of chains, no violation) were made by the author lane, by the independent review and by `diagnostic.py`. They corroborate Proposition 4.4 but are not used in its proof.

5. Non-embeddability

Proposition 5.1. There is no $\Phi : X \to \mathbb{R}$ such that $\delta(x, y) < \delta(z, w)$ implies $|\Phi(x) - \Phi(y)| < |\Phi(z) - \Phi(w)|$ for all x, y, z, w. In particular X does not embed in $\mathbb{R}^1$.

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Proof. Suppose Φ exists. Since $D(0,0) = 0 < D(0,5)$, we have $\Phi(0) \neq \Phi(5)$; replacing Φ by $-\Phi$, which preserves all distances, assume $\Phi(0) < \Phi(5)$. For $x \in \{1,2,3,4\}$ the table gives $D(0,x) < D(0,5)$ and $D(x,5) < D(0,5)$; hence $\Phi(x) - \Phi(0) < \Phi(5) - \Phi(0)$ and $\Phi(5) - \Phi(x) < \Phi(5) - \Phi(0)$, so $\Phi(0) < \Phi(x) < \Phi(5)$. Then $|\Phi(x) - \Phi(0)| = \Phi(x) - \Phi(0)$, and $D(0,1) < D(0,2) < D(0,3) < D(0,4)$ forces $\Phi(0) < \Phi(1) < \Phi(2) < \Phi(3) < \Phi(4) < \Phi(5)$. Put $g_i = \Phi(i) - \Phi(i-1) > 0$ for $i = 1, \ldots, 5$, so that $|\Phi(a) - \Phi(b)| = g_{a+1} + \cdots + g_b$ for $a < b$. The five comparisons in Table 2 hold in Table 1. Adding the five resulting inequalities gives $g_1 + 2g_2 + 2g_3 + 2g_4 + g_5 < g_1 + 2g_2 + 2g_3 + 2g_4 + g_5$, a contradiction. ∎

comparison	values	consequence
$D(2,3) < D(3,4)$	$1 < 2$	$g_3 < g_4$
$D(1,2) < D(2,4)$	$3 < 4$	$g_2 < g_3 + g_4$
$D(0,1) < D(1,3)$	$5 < 6$	$g_1 < g_2 + g_3$
$D(1,4) < D(4,5)$	$7 < 8$	$g_2 + g_3 + g_4 < g_5$
$D(3,5) < D(0,2)$	$9 < 10$	$g_4 + g_5 < g_1 + g_2$
sum		both sides: $g_1 + 2g_2 + 2g_3 + 2g_4 + g_5$

Table 2. The five-comparison certificate. The gap-coefficient vectors of the two sides both sum to (1, 2, 2, 2, 1).

Remark 5.2. Once the order of the points is forced, and since D has no ties, embeddability is the feasibility of a homogeneous system of strict linear inequalities in $g \in \mathbb{R}^5$ with $g > 0$. By Gordan's theorem of the alternative [8], infeasibility is always witnessed by a nonnegative combination of the inequalities that sums to zero; Table 2 is such a witness with five unit coefficients. Since the majorization property holds, this cancellation is not one of the majorization conditions: those compare the total intervals of two chains whose pieces can be matched termwise, while the five intervals on each side of Table 2 do not arise in that way.

6. Formal verification

The Lean module `OrdinalLine.lean` (20,774 bytes, sha256 `da892a4d…cec7703f`) is copied byte for byte from the author lane; it imports only Mathlib and is checked with Lean 4.34.1 (commit `5045d005`) and Mathlib `d13f23b7` [9]. Its statement of the conjecture and the two endpoints are:

```
def MajorizationConjecture : Prop :=
  ∀ (X : Type) (δ : X → X → X → X → Ordering) (n : ℕ) (e : Fin n → X),
    0 < n → IsOrdinalSpace δ → Function.Bijective e → HasMajorization δ e →
    EmbeddableInLine δ
theorem witness : IsOrdinalSpace δW ∧ Function.Bijective (id : Fin 6 → Fin 6) ∧
    HasMajorization δW (id : Fin 6 → Fin 6) ∧ ¬ EmbeddableInLine δW
theorem not_majorizationConjecture : ¬ MajorizationConjecture
```

source	Lean (namespace OrdinalSpaces366371)	remarks
Definition 2.1	`IsOrdinalSpace δ`	fields `ax_i`–`ax_vii`; $\delta : X \to X \to X \to X \to$ Ordering; (iii) via `Ordering.swap`; "$\in$ {'<', '='}" as $\neq$ `.gt`
Definition 2.2	`EmbeddableInLine δ`	$\exists\, \Phi : X \to \mathbb{R}$ with the three equivalences for $\lvert\Phi\, x - \Phi\, y\rvert$
$\mathscr{I}_n$, <, ~	`Prec δ e I J`, `Sim δ e I J`	I, J : Fin (k+1) → Fin n; one π : Equiv.Perm (Fin k); piece l is (I l.castSucc, I l.succ)
majorization property	`HasMajorization δ e`	for all k > 0 and all monotone I, J (repeats allowed)
Conjecture 2.4	`MajorizationConjecture`	$\lvert X\rvert = n$ encoded by a bijection e : Fin n → X; $x_{i+1} = e\ i$
Table 1	`D`, `δW`	δW x y z w = compare (D x y) (D z w)
Section 3	`isOrdinalSpace_W`	uses `D_symm`, `D_zero_iff` (kernel `decide`)
Lemma 4.1	`count_le`, `count_lt`, `card_eq_sum`	

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source	Lean (namespace OrdinalSpaces366371)	remarks
Lemma 4.2	`adj`, `dd`, `lastD`, `red`, `adj_dd`, `lastD_dd`, `pairwise_dd`, `red_mem_sublists`, `sum_pieces_eq_adj`, `adj_red`	recursion via `List.rec`, so no compiler auxiliaries
Lemma 4.3	`okB`, `finite_check`, `okB_spec`	`decide +kernel` over 64 × 64 sublists, t = 1, …, 15
Proposition 4.4	`majorization_W`	
Proposition 5.1	`core`, `not_embeddable_W`	`core` assumes Φ 0 < Φ 5; reflection in `not_embeddable_W`
Theorem 3.1	`witness`, `not_majorizationConjecture`	endpoints

Table 3. Correspondence between the paper and the Lean module. The conjecture is quantified over `X : Type`; its negation there already refutes the source statement, since the witness lives on `Fin 6`.

Replay. A source-only archive was extracted with a path-checking extractor into a new directory, and a portable driver (paths given by `LEAN_BIN` and `LEAN_PACKAGES`; it refuses pre-existing oleans) ran every step with at most 2 CPUs, 120 s CPU, 140 s wall, a 6 GiB private-memory guard and a systemd scope with MemoryMax=6G. The strict build `lean -j1 -DwarningAsError=true` produced an empty log, and its olean is byte-identical to the ones reported by the author lane and the independent review. `leanchecker` re-checked the module. A reviewer audit enumerated all 95 constants stored in the module (23 definitions, 69 theorems including generated auxiliaries, and the structure with its constructor and recursor), printed their full types, required a body for every definition and theorem, rejected axioms, opaque, unsafe, partial, implemented_by and extern constants and any `sorry`, and found the transitive axioms of every constant within {propext, Classical.choice, Quot.sound}. No `native_decide` or `ofReduceBool` is used.

Negative controls. Each asserts only what it tests: dropping the fifth comparison of Table 2 makes `linarith` fail; exchanging the values of the pairs {0, 3} and {0, 4} makes the kernel evaluate the finite check of Lemma 4.3 to `false` (a non-majorizing table is rejected); the endpoint does not inhabit the conjecture; `sorry` and an added axiom are rejected; a malformed olean is rejected by `leanchecker`. Two positive twins re-check the verbatim extracted fragments, validating the harness.

Certified content versus diagnostics. Theorem 3.1, Lemmas 4.1–4.3 and Propositions 4.4 and 5.1 are Lean-certified (Proposition 5.1 in the form ¬ EmbeddableInLine δW; Lemmas 4.1–4.2 in the forms listed in Table 3). Remarks 2.5 and 5.2 are ordinary mathematics and are not formalized. The following are diagnostics only, used by no proof: the literal checks for k ≤ 4 mentioned above; and the author lane's floating-point linear-programming scans, which report that every majorizing weak order on five points found by its pruned search (477, of which 114 tie-free) is embeddable, and that among the 2612 majorizing tie-free orders on six points it found, 4 are non-embeddable, including Table 1. These scans were not independently reproduced. We make no minimality claim; in particular we do not claim that Conjecture 2.4 holds for n = 5.

7. Relation to measurement theory and to citing work

Representing a finite comparison structure by numbers is a linear feasibility problem, so by the theorem of the alternative non-representability always has a cancellation certificate. Scott [2] and Krantz, Luce, Suppes and Tversky [3] develop this viewpoint: finite representability is characterized by cancellation conditions of all orders, and conditions of bounded order do not suffice in general. The classical instance closest in spirit to ours is the counterexample of Kraft, Pratt and Seidenberg [4] to de Finetti's axioms for qualitative probability.

Those results do not decide Conjecture 2.4. The majorization property is a family of conditions of *unbounded* order (every chain length k) but of a *restricted shape*: each side is a chain along the enumeration, the chain pieces are matched termwise by a permutation, and only the two total intervals are compared. A general statement that bounded-order conditions are insufficient says nothing about this particular unbounded family, so an explicit example was needed. Theorem 3.1 shows that the family misses a cancellation of order five.

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Two works citing [1] were checked: Pech and Pech, *Echeloned spaces* [5], read by the independent review (metrizability, Fraïssé limits and Ramsey properties; nothing on embeddings in the line or on majorization), and Stumme, Dürrschnabel and Hanika, *Towards ordinal data science* [6], identified by the review as a position paper; we did not read its body.

8. Limitations and status

- Source. We read arXiv:2412.17391v1 (submitted 23 December 2024). The journal version [1] was published in 2019/2020; its text was not read (institutional access only), and its Conjecture 7.4 is assumed to coincide with arXiv v1. MathDB cites the arXiv version.
- Prior art. Searches were bounded: one web search and one arXiv author search by the author lane, and five web searches by the independent review (one specific to this conjecture), all on 1 November 2026, found no refutation. The classical unidimensional-scaling and absolute-difference literature was not searched body by body. **We make no priority claim.**
- Significance. The witness was found by computer search and the certificate is short; the result settles the case left open in [1] but is modest in depth and expected in spirit.
- Trust. The Lean kernel, compiler and prebuilt Mathlib oleans are trusted (no upstream rebuild); statement fidelity rests on human reading of the source. This is a DRAFT with zero accepted credit; acceptance is reserved to the campaign root.

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