

From Ricci Flow to a Checked Proof

The Poincaré conjecture: the Hamilton–Perelman proof
and its Lean formalization, explained
(The Justin Sun Prize problem JSP-000007)

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Abstract

In 1904 Poincaré asked whether every closed, simply connected three-manifold is the three-sphere. Perelman answered yes in 2002–2003, completing Hamilton’s programme of deforming a metric by the Ricci flow and cutting it apart at singularities. In September 2026 Qin, Liao, Khaitan and Chow released a Lean 4 formalization of this Hamilton–Perelman proof [1]. On 27 October 2026, MathDB’s verifier checked an adapted version against its frozen problem statement and labelled it *Lean verified*.

This note explains both the mathematics and the formal artifact to a reader with a first course in differential geometry. We show how the Ricci flow rounds out a metric, why singularities are unavoidable, and how Perelman’s noncollapsing and canonical-neighbourhood theorems make surgery possible. We explain why a simply connected manifold must become extinct in finite time; that argument is short enough to check line by line, and we do so. We then show how extinction yields S^3 . The second half maps these steps onto the library that formalizes them and records exactly what the verification certifies. Every statement is tagged so the reader can see which claims come from the literature or the formalization, and which we computed ourselves.

Status (as of 29 October 2026)

- **Mathematics:** proved by Perelman [5, 6, 7], building on Hamilton. Detailed accounts by Kleiner–Lott [14], Morgan–Tian [15, 16] and Cao–Zhu [17] have been checked by the community. MathDB lists the problem as *Solved*.
- **Formalization:** Qin–Liao–Khaitan–Chow, DifferentialGeometry v0.1.3 [1, 2]. Their MathDB-adapted revision 9f65eaf9 received MathDB’s *Lean verified* label on 27 October 2026, with kernel replay and the axiom audit passed (§6.3).
- **Justin Sun Prize catalog:** at the time of writing it still records “Lean proof: No” for JSP-000007. It predates the MathDB verification.
- **This note:** exposition only. The author did not rebuild the Lean project.

Reading conventions. [Literature] marks results from the published literature. [Formalization] marks facts about the Lean development or its verification, taken from [1, 2, 3]. [Check] marks a computation we carried out ourselves. [Heuristic] marks an intuitive picture rather than a proof step.

*Independent. Version 1.0. This exposition was prepared with the assistance of an AI system (Claude, Anthropic). Statements about the formalization were checked against the cited paper, repository and MathDB verification record. Corrections are welcome.

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1 The question

Theorem 1 (Poincaré conjecture; Perelman). *Every closed (compact, boundaryless), simply connected topological three-manifold is homeomorphic to the three-sphere $S^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$.*

[Literature] Poincaré first asked whether trivial *homology* suffices, and then found the counterexample now called the Poincaré homology sphere. Its fundamental group is the binary icosahedral group of order 120. In 1904 he replaced homology by the fundamental group [4]. The higher-dimensional analogue, that a homotopy n -sphere is homeomorphic to S^n , was proved by Smale for $n \geq 5$ [19] and by Freedman for $n = 4$ [20]. Dimension three resisted every topological method and was finally settled by geometric analysis.

Smooth versus topological. By Moise's theorem [21], every topological three-manifold carries a smooth structure, unique up to diffeomorphism. The conjecture is therefore equivalent to its smooth form: *every closed, simply connected smooth three-manifold is diffeomorphic to S^3* . The formalization proves both forms, and Moise's theorem is part of it (§6).

Why geometry? [Heuristic] Thurston's geometrization conjecture predicts that every closed orientable three-manifold decomposes into pieces carrying one of eight model geometries. The simply connected case should carry the round geometry. Hamilton's idea was to *find* that geometry by letting a metric evolve until it becomes round, and to read off the topology from what the evolution does.

2 Ricci flow: heat flow for a metric

[Literature] Hamilton's Ricci flow [8] evolves a Riemannian metric $g(t)$ on M by

$$\partial_t g = -2 \operatorname{Ric}(g). \tag{1}$$

In harmonic coordinates, Ric is $-\frac{1}{2}\Delta g$ plus lower-order terms, so (1) is a heat equation for the metric. It is only weakly parabolic, because of diffeomorphism invariance. Short-time existence uses DeTurck's trick [13], which is also the route taken in the formalization [1]. The flow spreads curvature out: positively curved regions shrink and negatively curved regions expand.

Worked example ([Check] the round sphere). On the unit S^3 , $\text{Ric} = 2g_0$. Try $g(t) = r(t)^2 g_0$. Ricci curvature is scale-invariant, so $\text{Ric}(g(t)) = 2g_0$ and (1) becomes $(r^2)' = -4$. Hence $g(t) = (1 - 4t)g_0$: the sphere shrinks homothetically and becomes extinct at $t = 1/4$. This is the model for the whole proof. A simply connected manifold should eventually disappear in a round point.

Evolution of scalar curvature. [Literature] Under (1) the scalar curvature satisfies $\partial_t R = \Delta R + 2|\text{Ric}|^2$. In dimension three $|\text{Ric}|^2 \geq \frac{1}{3}R^2$, so

$$\frac{d}{dt}R_{\min} \geq \frac{2}{3}R_{\min}^2 \implies R \geq -\frac{3}{2(t+c)} \quad \text{if } R(\cdot, 0) \geq -\frac{3}{2c}. \quad (2)$$

[Check] The barrier $\varphi(t) = -3/(2(t+c))$ satisfies $\varphi' = 3/(2(t+c)^2) = \frac{2}{3}\varphi^2$, so the maximum principle keeps $R_{\min} \geq \varphi$. The formalization uses exactly this barrier and shows that it survives surgery [1, §2].

Hamilton's 1982 theorem. [Literature] If $\text{Ric} > 0$ initially, the normalized flow converges to a metric of constant positive curvature [8]. So M is a spherical space form S^3/Γ . In general, however, the flow develops *singularities*: curvature blows up in finite time, typically along *necks*, thin regions close to a round $S^2 \times$ interval. A neck pinch disconnects the manifold. The flow cannot simply be continued.

3 Perelman's three ingredients

Hamilton proposed to stop the flow just before a neck pinches, cut along the neck, cap the two ends with balls, and restart. Making this work needs a precise description of every high-curvature region. Perelman supplied it [5, 6].

3.1 No collapsing

[Literature] Perelman introduced the entropy

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-n/2} e^{-f} dV,$$

which is monotone under the flow when f and τ evolve suitably. From it he proved *κ -noncollapsing*: if $|\text{Rm}| \leq r^{-2}$ on a ball $B(x, r)$ with $r \leq r_0$, then $\text{vol } B(x, r) \geq \kappa r^3$ [5]. [Heuristic] Rescaled near a singularity, the flow cannot look like a thin, collapsed torus. This rules out the most dangerous singularity models that Hamilton could not exclude.

3.2 Pinching and κ -solutions

[Literature] The Hamilton–Ivey estimate [9, 12] says that where curvature is large, it is almost nonnegative. Blow-up limits at singularities are therefore *κ -solutions*: ancient solutions, defined for all $t \leq 0$, with bounded nonnegative curvature that are κ -noncollapsed. Perelman showed that these are compact in the pointed sense [10, 5] and described their structure. Apart from round space forms, every point of a κ -solution lies in a region that looks like a round neck $S^2 \times$ interval, or like a *cap*: a ball bounded by such a neck. The basic examples are the shrinking round sphere, the shrinking round cylinder $S^2 \times \mathbb{R}$ and its $\mathbb{Z}_2$ -quotient.

3.3 Canonical neighbourhoods

[Literature] The *canonical neighbourhood theorem* transfers this classification back to the flow [5, 6]. Every point of sufficiently high curvature has a neighbourhood that, after rescaling, is close to one of three models:

- an ε -neck: nearly a round $S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1})$;
- an ε -cap: a ball whose boundary region is an ε -neck;
- a closed component of positive curvature, which is diffeomorphic to a spherical space form.

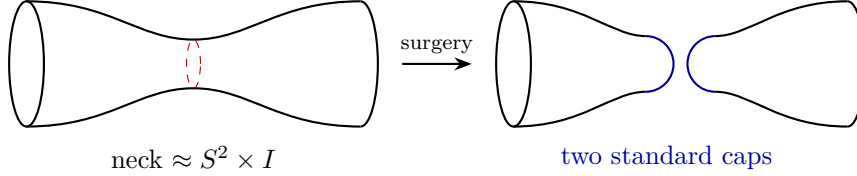


Figure 1: [Heuristic] Surgery: a thin neck is cut and both ends are capped with standard balls. Topologically this undoes a connected sum or removes an $S^2 \times S^1$ handle.

3.4 Ricci flow with surgery

[Literature] Using canonical neighbourhoods, Perelman performs surgery at a scale h far below the curvature scale, inside ε -necks [6]. Components that are entirely high-curvature, and are therefore of known topology, are discarded. Two facts make the process well defined:

- the canonical-neighbourhood and noncollapsing estimates *persist after surgery*, which requires carefully chosen surgery parameters;
- surgeries do not accumulate. [Check] By (2), the volume satisfies $\frac{d}{dt}V = -\int R dV \leq \frac{3}{2(t+c)}V$. So $V(t) \leq V(0)((t+c)/c)^{3/2}$ stays bounded on finite intervals, while each surgery removes a definite amount of volume. The formalization records this as a “uniform positive volume cost per cut on a fixed time horizon” [1, §2].

[Heuristic] Topologically, each surgery undoes a connected sum or removes an $S^2 \times S^1$ summand. Discarded pieces are spherical space forms, $S^2 \times S^1$, or connected sums of these. So the topology of M can be reconstructed from the pieces.

4 Finite extinction: why simple connectivity makes M disappear

For a general manifold, the flow with surgery may run forever, and that is where geometrization comes from. For a *simply connected* M , Perelman [7] and independently Colding–Minicozzi [18] showed that everything becomes extinct in finite time. The argument is short and quantitative, and we give it in the form used by the formalization [1, §2].

A topological width. [Formalization] Let $\Lambda_0 M$ be the space of contractible loops. If M is closed, oriented and simply connected, Poincaré duality and Hurewicz give $\pi_2(M) = 0$ and $\pi_3(M) \cong H_3(M; \mathbb{Z}) \cong \mathbb{Z}$. The fundamental class then determines a nontrivial homotopy class α_M of maps $S^2 \rightarrow \Lambda_0 M$. For a loop γ , let $A_g(\gamma)$ be the least area of a disk it bounds, and define the width

$$W(M, g) = \inf_{[\Gamma] = \alpha_M} \sup_{z \in S^2} A_g(\Gamma(z)).$$

In Lean this is `canonicalWidth`, and the class is `positiveFreeContractibleClass`. Because α_M is nontrivial, $W > 0$.

The key inequality. [Literature] [Formalization] Hamilton’s area estimate [11] and Perelman’s argument [7] give, between surgeries,

$$D^+W(t) \leq -2\pi - \frac{1}{2}R_{\min}(t)W(t) \leq -2\pi + \frac{3}{4(t+c)}W(t). \quad (3)$$

The second inequality is (2). [Heuristic] The constant -2π comes from the Gauss–Bonnet theorem applied to a minimal disk. At surgery times the width does not jump up [1, §2].

Worked example ([Check] integrating (3)). Multiply by the integrating factor $(t+c)^{-3/4}$:

$$\frac{d}{dt} \left[\frac{W(t)}{(t+c)^{3/4}} \right] \leq -\frac{2\pi}{(t+c)^{3/4}} \implies 0 \leq \frac{W(t)}{(t+c)^{3/4}} \leq \frac{W_0}{c^{3/4}} - 8\pi((t+c)^{1/4} - c^{1/4}),$$

using $\int_0^t 2\pi(s+c)^{-3/4} ds = 8\pi((t+c)^{1/4} - c^{1/4})$. The right-hand side becomes negative after

$$T_* = \left(c^{1/4} + \frac{W_0}{8\pi c^{3/4}} \right)^4 - c.$$

No component can survive past T_* . This matches `extinctionThreshold` in [1, §2].

Finite extinction

For a closed, oriented, simply connected (M, g_0) , Ricci flow with surgery reaches the empty manifold at a finite time. Every component discarded along the way is a connected sum of spherical space forms and copies of $S^2 \times S^1$. In the formalization this is `exists_poincare_controlled_extinction`.

5 From extinction to the sphere

[Literature] [Formalization] Reverse the surgery history. Each surgery replaced M by pieces whose connected sum gives back the earlier manifold, and the discarded pieces are of the known types. So

$$M \cong (S^3/\Gamma_1) \# \cdots \# (S^3/\Gamma_k) \# (S^2 \times S^1) \# \cdots \# (S^2 \times S^1).$$

By van Kampen, $\pi_1(M)$ is the free product $\Gamma_1 * \cdots * \Gamma_k * \mathbb{Z} * \cdots * \mathbb{Z}$. If $\pi_1(M) = 1$, there are no $S^2 \times S^1$ summands, since each contributes $\mathbb{Z}$, and every Γ_i is trivial. So every summand is S^3 , and $M \cong S^3$.

[Check] Examples of excluded summands: $S^2 \times S^1$ has $\pi_1 = \mathbb{Z}$, the lens space $L(p, q)$ has $\pi_1 = \mathbb{Z}/p$, and the Poincaré homology sphere has π_1 of order 120. Simple connectivity alone rules out all of them.

Back to topology. For a merely topological M , Moise’s theorem provides a compatible smooth structure. The smooth theorem gives a diffeomorphism to S^3 , and its underlying map is the required homeomorphism. Simple connectivity also implies orientability, which the extinction argument uses.

6 The Lean formalization

6.1 The endpoints

[Formalization] Qin–Liao–Khaitan–Chow [1] prove three main theorems in Lean 4 with Mathlib (release v0.1.3, commit 7a48598d, Lean/Mathlib v4.33.1):

- **Moise smoothing:** every closed topological three-manifold admits a compatible C^∞ structure (`exists_isManifold_three`);
- **smooth Poincaré:** every simply connected closed smooth three-manifold is diffeomorphic to S^3 (`smoothPoincareConjecture_holds`);
- **topological Poincaré:** `DifferentialGeometry.Topology.poincare_conjecture`. For a compact, Hausdorff, simply connected space M charted on $\mathbb{R}^3$, it gives

`Nonempty (M  $\simeq_t$  Metric.sphere (0 : EuclideanSpace  $\mathbb{R}$  (Fin 4)) 1).`

The paper notes that this proves the three-dimensional Poincaré statement that Mathlib marks `proof_wanted`. It also reports that `#print axioms` for all four main endpoints, the three above and the extinction theorem, returns only `propext`, `Classical.choice` and `Quot.sound`.

6.2 What it took

[Formalization] At the verified revision, the library contains about 17,600 Lean files (about 240 MB of source) under `DifferentialGeometry/`. The Poincaré import closure spans 14,234 modules, and the authors’ audit reports about 237,000 declarations in it [2]. The largest areas give a map from mathematics to code. File counts come from the Git tree; the roles are inferred from directory and module names:

Library area	Files	Mathematical role
Geometry/Flow	4,341	Ricci flow and its estimates, surgery and extinction
Topology/PiecewiseLinear	2,769	triangulations and PL approximation for Moise smoothing
Analysis/Parabolic, Spectral	1,467	parabolic and spectral analysis used by the flow estimates
Analysis/Sobolev, Elliptic	1,025	function spaces, elliptic regularity (includes adapted De Giorgi theory)
Geometry/Compactness, Comparison	830	Cheeger–Gromov compactness, comparison geometry
Topology/ThreeManifold, Homology	750	connected sums, van Kampen, Hurewicz, the final endpoint

The paper credits adapted developments by others: De Giorgi–Nash–Moser theory (Armstrong–Kempe), Jordan–Schoenflies (Begué), surface classification (SF Lean meetup), Sard and Morse arguments (Tau Ceti), and the Riemann mapping theorem (Kudryashov). It also states that automated assistance was used for proofs and prose [1].

6.3 What MathDB verified

[Formalization] MathDB states the problem in Lean as a frozen target, `FtpEvalBench.YPoincare_poincare_conjecture`, whose hypothesis is a homotopy equivalence $M \simeq S^3$. The submitted `Submission.lean` is a short adapter. It derives compactness and simple connectivity of M from the homotopy equivalence, then applies `poincare_conjecture`. The public MathDB record for revision 9f65eaf9 of `liao9yuan/differential-geometry-mathdb` shows [3]:

- outcome *verified*, label *Lean verified*, completed 27 October 2026 at 20:32 UTC, after about 89 minutes of grading;
- passed stages: interface, kernel, replay (all runtime modules), dependencies and axioms; the axiom audit reported no violations;
- reason: “kernel checked the frozen target; axiom audit passed”.

What verification does and does not mean

The Lean kernel has checked that the frozen target follows from Lean’s foundations, Mathlib and the submitted library. It used no `sorry` and only the three standard axioms. That is far stronger than a refereed paper on the question of *logical* correctness. Two human questions remain. First, does the frozen target say what Poincaré asked? A homotopy equivalence with S^3 is equivalent, in dimension three, to being closed and simply connected; the adapter proves the direction it needs. Second, the definitions used inside the library (manifolds, Ricci flow, surgery) matter only through the final statement, which mentions none of them. That is the main advantage of a formal proof: a reader only has to trust a statement a few lines long.

7 Why this matters

- **Scale.** The Poincaré conjecture is the one Millennium Prize Problem whose solution was accepted long before 2026. It is now machine-checked, together with the analytic machinery (parabolic PDE, compactness, surgery) that was long considered out of reach for formalization.
- **Trust.** Perelman’s proof was checked by several teams over years, and one detailed account needed a published correction [16]. A kernel-checked proof makes the logical question final, given the statement.
- **Reuse.** Ricci flow, Cheeger–Gromov compactness and PL topology are now available in Lean, which opens the way to formalizing geometrization and further geometric analysis.

A A map from the proof to the formal endpoints

Mathematical step	Lean name or location [1]
Short-time existence (DeTurck)	<code>Geometry/Flow/RicciFlow/...</code>
Scalar-curvature barrier (2), preserved by surgery	§2 of [1]
Noncollapsing, κ -solution compactness, canonical neighbourhoods	“main interfaces” in [1, §2]
Surgery: neck cuts, caps, restarts, volume cost	<code>Geometry/Flow/RicciFlow/Surgery/...</code>
Topological width W and class α_M	<code>canonicalWidth, positiveFreeContractibleClass</code>
Width along the surgery history; extinction time	<code>historyWidth, extinctionThreshold</code>
Finite extinction	<code>exists_poincare_controlled_extinction</code>
Smooth Poincaré	<code>smoothPoincareConjecture_holds</code>
Moise smoothing	<code>exists_isManifold_three</code>
Topological Poincaré	<code>DifferentialGeometry.Topology.poincare_conjecture</code>
MathDB frozen target	<code>Submission.solution : Problem.Target</code>

Originality and attribution

The mathematics is due to Poincaré, Hamilton, Perelman and the authors of the cited accounts. The formalization is due to Qin, Liao, Khaitan and Chow, and to the developers they credit. The verification is MathDB’s. This note contributes only exposition: the organization, the explanations, the computations marked [Check] (the round-sphere extinction time, the scalar-curvature barrier, the volume bound and the integration of the width inequality), the map from proof steps to formal endpoints, and the reading of the verification record.

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