

A two-term counterexample to Bessel–Fourier kernel positivity

Research proof and Lean formalization

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Abstract

We refute Acosta Babb’s positivity conjecture for a modified Fejér kernel of Bessel–Fourier series. At the strictly interior parameters $\alpha = 2/5$ and $x = 47/50$, the two-term kernel is less than $-1/500$. Exact rational series bounds identify the genuine first two positive Bessel zeros and certify the negative value. The result is formally proved in Lean without numerical root-location assumptions.

Let j_m be the m th positive zero of the Bessel function J_0 . Acosta Babb defines the order-zero kernel

$$K_M^\alpha(x) = \sum_{m=1}^M \left(1 - \frac{m}{M+1}\right) j_m^\alpha J_0(j_m x) \quad (1)$$

and conjectures that it is nonnegative whenever $0 \leq \alpha \leq 1/2$, $M \geq 1$, and $0 \leq x \leq 1$ [1, Conjecture 1, p. 11]. The same assertion appears in the earlier preprint [2, Conjecture 1, p. 10]. The published discussion fixes Bessel order zero, which is the case used here.

Theorem 1. *For the kernel (1),*

$$K_2^{2/5}(47/50) < -\frac{1}{500}.$$

In particular, the conjectured nonnegativity fails with both parameters strictly inside their stated intervals.

Series and uniform error bounds. Put $q = x^2/4$ and write

$$J_0(x) = \sum_{k \geq 0} \frac{(-1)^k q^k}{(k!)^2}, \quad H(q) = \sum_{k \geq 0} \frac{(-1)^k q^k}{k!(k+1)!}.$$

Normal convergence on compact sets justifies

$$J'_0(x) = -(x/2)H(x^2/4).$$

Let $p(q)$ and $h(q)$ be the respective sums for $0 \leq k < 20$. For $0 \leq q \leq 9$, the absolute ratio of successive terms in either tail, starting at index 20, is at most $9/21^2 < 1/2$. Therefore

$$\begin{aligned} |J_0(x) - p(x^2/4)| &\leq e := \frac{2 \cdot 9^{20}}{(20!)^2} < 10^{-17}, \\ |H(q) - h(q)| &\leq e_H := \frac{2 \cdot 9^{20}}{20! 21!} < 10^{-18}. \end{aligned} \quad (2)$$

The first bound applies for $0 \leq x \leq 6$.

Finite rational certificates. All terminating decimals below denote exact rational numbers. Set

$$(\ell_1, u_1) = (2.4048, 2.4049), \quad (\ell_2, u_2) = (5.5200, 5.5201), \quad c = 47/50.$$

The following polynomial bounds hold throughout the indicated intervals:

Polynomial	Interval for q	Bound
$h(q)$	$[0, 9/4]$	$h(q) \geq 1/5$
$p(q)$	$[9/4, \ell_2^2/4]$	$p(q) \leq -1/50000$
$h(q)$	$[\ell_2^2/4, u_2^2/4]$	$h(q) \leq -3/25$
$p(q)$	$[(c\ell_1)^2/4, (cu_1)^2/4]$	$p(q) \leq 3081/40000$
$p(q)$	$[(c\ell_2)^2/4, (cu_2)^2/4]$	$p(q) \leq -11381/100000$

For completeness, these are exact coefficient checks in the Bernstein basis. If $P(q) = \sum_{r=0}^{19} a_r q^r$ on $[u, v]$, define

$$\beta_i = \sum_{k=0}^i \frac{\binom{i}{k}}{\binom{19}{k}} (v-u)^k \sum_{r=k}^{19} a_r \binom{r}{k} u^{r-k}. \quad (3)$$

Then

$$P(u + (v-u)t) = \sum_{i=0}^{19} \beta_i \binom{19}{i} t^i (1-t)^{19-i}.$$

For $0 \leq t \leq 1$ the basis functions are nonnegative and sum to one. Substitution of the displayed coefficients of p and h into (3) gives the five bounds in the table by comparing each β_i to its stated bound. All coefficient comparisons and polynomial identities are checked in the Lean certificate; no sampling is used.

Identification of the first two zeros. Direct rational evaluation of p , together with (2), gives

$$J_0(\ell_1) > 0, \quad J_0(u_1) < 0, \quad J_0(\ell_2) < 0, \quad J_0(u_2) > 0. \quad (4)$$

The first table row and (2) imply $H(q) > 0$ on $[0, 9/4]$. Consequently J_0 is strictly decreasing on $[0, 3]$, and (4) identifies its unique zero there in $[\ell_1, u_1]$. The second row implies $J_0(x) < 0$ for $3 \leq x \leq \ell_2$, so there are no further zeros in that interval. The third row implies $H(x^2/4) < 0$ on $[\ell_2, u_2]$, where J_0 is strictly increasing. Its endpoint signs give exactly one zero there. Thus

$$j_1 \in [\ell_1, u_1], \quad j_2 \in [\ell_2, u_2]. \quad (5)$$

This proves the zero indices as well as their enclosures.

Proof of Theorem 1. Positive fifth powers are strictly increasing. The exact rational inequalities

$$1.4204^5 < \ell_1^2, \quad u_1^2 < 1.4205^5, \quad 1.9805^5 < \ell_2^2, \quad u_2^2 < 1.9806^5$$

and $(j_i^{2/5})^5 = j_i^2$ give

$$1.4204 \leq j_1^{2/5} \leq 1.4205, \quad 1.9805 \leq j_2^{2/5} \leq 1.9806.$$

The last two table rows, (5), and (2) yield

$$J_0(cj_1) \leq \frac{7703}{100000}, \quad J_0(cj_2) \leq -\frac{569}{5000}.$$

Using positivity of the power factors and the negative sign of the second upper bound, we obtain

$$\begin{aligned} K_2^{2/5}(c) &\leq \frac{2}{3} \frac{14205}{10000} \frac{7703}{100000} - \frac{1}{3} \frac{19805}{10000} \frac{569}{5000} \\ &= -\frac{653867}{300000000} < -\frac{1}{500}. \end{aligned}$$

□

Formal scope and status. The certificate contains 87 declarations in eight modules. It proves actual infinite-series convergence, the derivative identity, uniform tails, whole-interval polynomial bounds, the first-two-positive-zero identification, and the literal real-power kernel inequality. The closed endpoint is `BesselFejer.source_counterexample_365454`. A fresh isolated parent replay passes with Lean 4.34.1 and mathlib revision

d13f23b723b8a846827a245b89c10fc7d3f11612,

using only `propext`, `Classical.choice`, and `Quot.sound`. Independent final source, mathematical, formal, manuscript, bounded novelty and substantive-eligibility review passes. Acceptance additionally requires a fresh replay of the extracted proof archive. No worldwide priority is claimed. This is one proposed conjecture refutation; the separate slow-decay existence conjectures and conditional consequences in the source are not decided here.

References

- [1] R. L. Acosta Babb, *On the rate of growth of Bessel–Fourier coefficients for integrable functions*, J. Math. Anal. Appl. **559** (2026), 130440. doi:10.1016/j.jmaa.2026.130440.
- [2] R. L. Acosta Babb, *A partial converse to the Riemann–Lebesgue lemma for Bessel–Fourier series of order zero*, arXiv:2410.17681v1 (2024).