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An injective C*-power series violating the Bieberbach coefficient bound

Research proof and Lean formalization

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Abstract

We give a normalized power series on $C([0,1],\mathbb{C})$ which converges in norm at every element and is globally injective, but whose second coefficient violates the proposed C*-algebraic Bieberbach order bound. Its formal derivative series also converges everywhere. Injectivity follows from a continuous exponential period anchored at zero; injectivity of individual scalar fibers is not assumed.

Krishna's Conjecture 1.2 in [1] proposes the following extension of the scalar Bieberbach theorem. If $\mathcal{A}$ is a unital C*-algebra and $F(u) = u + \sum_{n\geq 2} a_n u^n$ converges in norm and is injective on $\{u : \|u\| < 1\}$, then

$$a_n a_n^* \leq n^2 1, \qquad a_n^* a_n \leq n^2 1 \qquad (n \geq 2). \tag{1}$$

Conjecture 1.4 adds norm convergence of the derivative series. The next example violates both general-coefficient assertions. The separate invertible-coefficient variants are not claimed refuted.

Theorem 1. *Let $\mathcal{A} = C([0,1],\mathbb{C})$ with its supremum norm, and let $a(t) = 8t$. Put*

$$c_n = \frac{a^{n-1}}{n!} \quad (n \geq 1), \qquad F(u) = \sum_{n\geq 1} c_n u^n. \tag{2}$$

This series and its formal derivative series converge in the norm of $\mathcal{A}$ for every $u \in \mathcal{A}$. The map $F : \mathcal{A} \to \mathcal{A}$ is injective. Nevertheless $c_2(1) = 4$, so both inequalities in (1) fail at $n = 2$.

Proof. For every $u \in \mathcal{A}$,

$$\sum_{n\geq 1} \|c_n u^n\| \leq \|u\| \sum_{k\geq 0} \frac{\|au\|^k}{(k+1)!} < \infty.$$

The derivative series is dominated by an exponential series as well. Multiplication of (2) by a and reindexing the absolutely convergent exponential series gives the identity in $\mathcal{A}$

$$aF(u) = \exp(au) - 1. \tag{3}$$

Here the exponential is the Banach-algebra exponential, whose scalar evaluation is the usual complex exponential. Evaluation of the norm-convergent series at $t = 0$ also gives $F(u)(0) = u(0)$. In particular,

$$F(u)(t) = \frac{\exp(8tu(t)) - 1}{8t} \quad (t > 0).$$

Suppose $F(u) = F(v)$. By (3), the continuous function $h(t) = 8t(u(t) - v(t))$ satisfies $\exp(h(t)) = 1$ for every t, and $h(0) = 0$. Thus its image lies in the discrete set $2\pi i\mathbb{Z}$. Connectedness of $[0,1]$

forces $h = 0$. Since $8t \neq 0$ for $t > 0$, we obtain $u(t) = v(t)$ there, and the equality at 0 follows from $F(u)(0) = F(v)(0)$. Hence $u = v$.

Finally $c_1 = 1$ and $c_2(t) = 4t$. Evaluation of either claimed order bound at $t = 1$ would give $16 \leq 4$, a contradiction. □

Why the scalar theorem does not apply. The source requires injectivity of a map on the unit ball of the algebra. It does not require that every scalar fiber be univalent. A collision of two continuous inputs in this example would produce a continuous integer-valued exponential period, which must vanish because it vanishes at the endpoint. Some individual scalar fibers are noninjective. The classical scalar Bieberbach theorem is therefore not contradicted. The coefficients c_n for $n \geq 2$ vanish at 0 and are not invertible elements of $\mathcal{A}$; the source's invertible-coefficient restrictions cannot be silently discarded.

Formal scope and review status. The Lean certificate has 28 declarations in four modules. It proves actual Banach-norm summability, the exponential identity, scalar evaluation, global injectivity, the literal coefficient-order violation, and closed negations of the two source assertions specialized to the genuine commutative unital C*-algebra $C([0, 1], \mathbb{C})$. No convergence, injectivity or fiber-choice hypothesis remains. The endpoints are

`source_counterexample`, `source_strong_counterexample`

in namespace `CStarBieberbach`. The isolated strict replay uses Lean 4.34.1 and mathlib revision

`d13f23b723b8a846827a245b89c10fc7d3f11612`,

with only `propext`, `Classical.choice` and `Quot.sound`. It checks the fixed coefficient $8t$; no separately formalized theorem about all parameter values is claimed.

Independent final review passes all five gates, with bounded novelty coverage.

The source was read in the author-deposited Zenodo version (doi:10.5281/zenodo.6917925), whose text agrees with the source reading of Preprints version 1. Direct Preprints PDF retrieval remains unavailable. Acceptance additionally requires a fresh replay of the extracted proof archive. The two variants use one construction and constitute one outcome. Classical exponential-series and connectedness facts and helper declarations are not additional discoveries. No worldwide priority claim is made.

References

[1] K. Mahesh Krishna, *C*-algebraic Bieberbach, Robertson, Lebedev–Milin, Zalcman, Krzyz and Corona Conjectures*, Preprints (2022), version 1, Conjectures 1.2 and 1.4, pp. 2–3. doi:10.20944/preprints202208.0140.v1.

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