Partition coefficients for generalized peakon transfer matrices

Research proof and Lean formalization

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Abstract

We give all partition coefficients in the trace powers of a generalized peakon Lax matrix. A rank-two determinant calculation reduces its characteristic determinant to one Hamiltonian generating polynomial. The formal trace-log identity then gives a closed factorial formula, including the two coefficient families proposed by Avan, Frappat and Ragoucy. The proof applies to arbitrary coherent complex particle amplitudes and, literally, to the square-root matrices for nonnegative real momenta.

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Scope and review status. The mathematical and formal statements below use the explicitly stated amplitude domains. Independent preliminary source and mathematical review confirms the coherent square-root convention used below. Independent final source, mathematical, formal, manuscript and bounded novelty review has passed. No worldwide priority claim is made.

Let $c_i \in \mathbb{C}$, $q_i \in \mathbb{R}$ $(1 \le i \le N)$, and $\nu, \gamma, z \in \mathbb{C}$, with $2z + \gamma \neq 0$. Define the actual hyperbolic matrices

$$A_{ij} = c_i c_j \sinh\Big(\frac{\nu}{2}(q_i - q_j)\Big), \qquad T_{ij} = c_i c_j \cosh\Big(\frac{\nu}{2}(q_i - q_j)\Big), \qquad S_{ij} = \operatorname{sgn}(q_i - q_j) A_{ij},$$
$$\Lambda = \frac{z^2 - 1}{2z + \gamma}, \qquad \xi = \frac{z^2 + \gamma z + 1}{2z + \gamma}, \qquad L = S + \Lambda T + \xi A.$$

Here $\operatorname{sgn}(0) = 0$. Put $\tau_m = \operatorname{tr}(L^m)$ and, exactly as in [1, eq. (5.17)],

$$h^{(k)} = k \operatorname{tr}\Big((T + \tfrac{\gamma}{2}A)(S + A)^{k-1}\Big) \qquad (k \ge 1). \tag{1}$$

Theorem 1. *For every $m \ge 1$,*

$$\tau_m = \sum_{j=1}^{m} \Lambda^j \sum_{\substack{k_1 \ge \cdots \ge k_j \ge 1 \\ k_1 + \cdots + k_j = m}} \frac{m(j-1)!}{(\prod_{i=1}^{j} k_i)(\prod_{s \ge 1} r_s!)} \prod_{i=1}^{j} h^{(k_i)}, \tag{2}$$

where $r_s = \#\{i : k_i = s\}$. In particular, the proposed coefficient families are

$$a_1(m) = 1, \qquad a_m(1, \dots, 1) = 1, \qquad a_{j+1}(m - j, \underbrace{1, \dots, 1}_{j}) = \frac{m}{m - j} \quad (0 \le j < m - 1). \tag{3}$$

No independence or uniqueness of these monomials for fixed N is asserted; the displayed universal coefficients give a valid expansion for every N.

Corollary 1. *For arbitrary $p_i \ge 0$, Theorem 1 holds for the literal factors $\sqrt{p_i p_j}$ in place of $c_i c_j$, with the same $h^{(k)}$ and all real particle positions, including ties.*

Indeed choose $c_i = \sqrt{p_i}$; then $\sqrt{p_i p_j} = \sqrt{p_i}\sqrt{p_j}$.

The determinant reduction

First reorder the positions so that $q_1 \geq \cdots \geq q_N$, and set

$$u_i = c_i e^{\nu q_i/2}, \qquad v_i = c_i e^{-\nu q_i/2}, \qquad B = S + A.$$

Expanding sinh and cosh gives

$$A = \tfrac{1}{2}(uv^t - vu^t), \qquad T = \tfrac{1}{2}(uv^t + vu^t), \qquad B_{ij} = \begin{cases} u_i v_j - v_i u_j, & i < j, \\ 0, & i \geq j. \end{cases}$$

These identities remain valid when positions tie, since then $A_{ij} = 0$. Thus $B^N = 0$, $\det(I - tB) = 1$, and

$$R = (I - tB)^{-1} = \sum_{k=0}^{N-1} t^k B^k, \qquad B - B^t = uv^t - vu^t.$$

All equalities are polynomial identities over a characteristic-zero field. Write

$$a = \frac{\Lambda + \xi - 1}{2}, \quad b = \frac{\Lambda - \xi + 1}{2}, \quad L = B + auv^t + bvu^t,$$

and put $P = v^t R u$, $Q = u^t R v$, $C = PQ - (v^t R v)(u^t R u)$. The classical rank-two determinant lemma gives

$$\det(I - tL) = 1 - t(aP + bQ) + t^2 abC. \tag{4}$$

The resolvent identity

$$R - R^t = tR(B - B^t)R^t$$

gives $P - Q = tC$ after multiplication by v^t and u. Also

$$\xi^2 = \Lambda^2 + \gamma\Lambda + 1, \qquad a + b = \Lambda, \qquad ab = \frac{\xi - 1}{2} - \frac{\gamma\Lambda}{4}.$$

Substitution in (4) cancels the $\xi - 1$ terms and proves

$$\det(I - tL) = 1 - \Lambda H(t), \qquad H(t) = t \operatorname{tr}\left((T + \tfrac{\gamma}{2}A)R\right) = \sum_{k=1}^{N} \frac{h^{(k)}}{k} t^k. \tag{5}$$

This is the source-specific reduction. All determinant, trace and Hamiltonian expressions are invariant under simultaneous permutation of the particle indices, so (5) holds without an ordering assumption. Nilpotence also proves $h^{(k)} = 0$ for $k > N$.

Every trace order and every partition

The classical formal trace-log identity, with constant determinant coefficient one, is

$$-\log\det(I - tL) = \sum_{m \geq 1} \frac{\operatorname{tr}(L^m)}{m} t^m.$$

For clarity, it can be proved by constructing the matrix geometric series $(I - tL)^{-1}$ and applying Jacobi's determinant derivative identity. Both sides then have the same derivative and zero constant term. Combining this identity with (5), using $H(0) = 0$, yields

$$\sum_{m \geq 1} \frac{\tau_m}{m} t^m = \sum_{j \geq 1} \frac{\Lambda^j}{j} H(t)^j. \tag{6}$$

Only $j \leq m$ contributes to the coefficient of t^m. A fixed partition with multiplicities r_s has $j!/\prod_s r_s!$ orderings in $H(t)^j$. Its contribution is therefore

$$\frac{m}{j} \frac{j!}{\prod_s r_s!} \prod_{i=1}^{j} \frac{h^{(k_i)}}{k_i},$$

which proves (2). The three evaluations in (3) follow by direct cancellation. The restriction $j < m-1$ ensures $m - j > 1$; the all-ones case has its own multiplicity factor.

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Source relation and formal scope

The Lax matrices originate in [3, 2]. Avan–Frappat–Ragoucy [1, Conjecture 5.2, eqs. (5.20)–(5.21)] propose the partition expansion and the families (3), leaving the other coefficients undetermined. Theorem 1 supplies all coefficients for the coherent-amplitude family; Corollary 1 uses their displayed square-root factors literally for $p_i \geq 0$. The source's notation $\sqrt{p_i p_j}$ uses coherent per-particle roots: its identity $T_{ii} = p_i$, from eqs. (2.2) and (5.6), and its degree-m homogeneity require this convention. Thus $c_i^2 = p_i$ recovers its family, including complex coherent roots for negative momenta. Arbitrary c_i in Theorem 1 covers this convention; the corollary additionally proves the nonnegative principal-root reading directly. The rank-two determinant lemma, Jacobi identity, formal trace-log identity and multinomial theorem are classical ingredients, not separate discoveries.

The Lean development has 121 declarations in 19 modules, using Lean 4.34.1 and mathlib revision
`d13f23b723b8a846827a245b89c10fc7d3f11612`. It proves the determinant collapse, the formal trace-log and multinomial bridges, all-order support truncation, the actual complex hyperbolic matrix identities, permutation sorting, and both coefficient families. The closed source endpoint and its literal nonnegative specialization are:

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source_conjecture_372008
nonnegative_source_conjecture_372008
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The isolated strict replay passes with only `propext`, `Classical.choice` and `Quot.sound`. Independent final review passes all five gates. Discovery acceptance additionally requires a fresh replay from the extracted proof packet; no global priority or optimality claim is made.

References

[1] J. Avan, L. Frappat and E. Ragoucy, *Quadratic Poisson brackets for the Camassa–Holm peakons*, J. Phys. A: Math. Theor. **59** (2026), 245202. doi:10.1088/1751-8121/ae7683; arXiv:2512.11066v3.

[2] J. Avan, L. Frappat and E. Ragoucy, *A new integrable structure associated to the Camassa–Holm peakons*, SciPost Phys. **15** (2023), 228. doi:10.21468/SciPostPhys.15.6.228.

[3] O. Ragnisco and M. Bruschi, *Peakons, R-matrix and Toda lattice*, arXiv:solv-int/9509012v1 (1995).

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