A Gaussian envelope for Hermite polynomials

Research proof and Lean formalization

October 27, 2026

Abstract

We prove the Hermite-envelope conjecture of Guo and Liu. For odd degrees, the initial oscillator energy is slightly larger than the desired coefficient permits. A short sine comparison gives enough strict energy loss to remove this excess. A real-root moment inequality excludes nonzero stationary peaks after the conjectured cutoff and supplies the clamped tail. The proof uses classical oscillator, Wallis and polynomial tools; no new general machinery or priority claim is made.

mathdb:2610.00789v1 28 Oct 2026

Verification. The literal source theorem is proved in Lean and has passed independent source, mathematical, formal, manuscript and bounded prior-art review. Reproducible proof files accompany this manuscript.

Let $H_n(x) = (-1)^n e^{x^2}(d/dx)^n e^{-x^2}$ be the physicists' Hermite polynomial, and set

$$E = 2n+1, \qquad C_n = \frac{2^n\Gamma((n+1)/2)}{\sqrt{\pi}}, \qquad a_n = \sqrt{E}\left(1 - \frac{\pi}{2E}\right), \qquad g_n(t) = \left(1 - \frac{t^2}{E}\right)^{-1/2}.$$

The cutoff satisfies $0 < a_n < \sqrt{E}$ for $n \geq 1$. Guo–Liu's Conjecture 1, published as [1, eq. (29), p. 8], is exactly the following statement.

Theorem 1. *For every integer $n \geq 1$ and real $x > 0$,*

$$|H_n(x)e^{-x^2}| \leq C_n g_n(\min\{x, a_n\})e^{-x^2/2}.$$

The bulk estimate

Write $y(x) = H_n(x)e^{-x^2/2}$. The Hermite equation gives

$$y'' + (E - x^2)y = 0, \quad B(x) = (E-x^2)y(x)^2 + y'(x)^2, \quad B'(x) = -2xy(x)^2. \tag{1}$$

For even n, $y'(0) = 0$ and $|y(0)| = C_n$, so $B(x) \leq EC_n^2$ on $x \geq 0$. This proves $|y(x)| \leq C_n g_n(x)$ for $0 \leq x < \sqrt{E}$ in every even degree.

For $n = 2m+1$, replace y by $(-1)^m y$, so $y(0) = 0$ and $y'(0) = A = 2(2m+1)(2m)!/m! > 0$. Put $\gamma_m = A^2/(EC_n^2)$. The exact initial values and the Wallis limit give

$$\gamma_m = \frac{\pi(2m+1)^2\binom{2m}{m}^2}{(4m+3)16^m}, \qquad \frac{\gamma_{m+1}}{\gamma_m} = 1 - \frac{1}{4(m+1)^2(4m+7)}, \qquad \gamma_m \longrightarrow 1. \tag{2}$$

For $\delta_m = \gamma_m(1 - 1/(2(4m+3)^2))$, direct cancellation yields

$$\delta_{m+1} - \delta_m = \gamma_m \frac{176m^2 + 440m + 247}{8(m+1)^2(4m+3)^2(4m+7)^3} > 0.$$

Since $\delta_m \to 1$, we have $\delta_m < 1$. For $m \geq 1$, also $\gamma_m \leq \gamma_1 = 9\pi/28 < 99/98$. Thus

$$A^2\left(1 - \frac{1}{2E^2}\right) < EC_n^2, \qquad A^2 \leq \frac{99}{98}EC_n^2 \quad (m \geq 1). \tag{3}$$

Assume now $m \geq 1$, so $E \geq 7$. Put $x_0 = \sqrt{2/E}$, $q_0 = E - 2/E$ and $c = \sqrt{1 - 2/E^2}$. On $0 \leq x \leq x_0$,

$$\frac{A}{\sqrt{E}} \sin(\sqrt{E}x) \leq y(x) \leq \frac{A}{\sqrt{q_0}} \sin(\sqrt{q_0}x). \tag{4}$$

To justify positivity as well as comparison, $y'(0) > 0$ first gives $y > 0$ near zero. If its first positive zero z were at most x_0, variation of constants would give

$$y(z) = \frac{A}{\sqrt{E}} \sin(\sqrt{E}z) + \int_0^z \frac{\sin(\sqrt{E}(z-t))}{\sqrt{E}}\, t^2 y(t)\, dt > 0,$$

because $\sqrt{E}z \leq \sqrt{2} < \pi$. This is a contradiction. The same identity proves the lower bound in (4). Applying variation of constants to $y'' + q_0 y = -(x_0^2 - x^2)y$ proves the upper bound; its sine kernel is nonnegative on this interval.

For $0 \leq u \leq \sqrt{2}$, set $M(u) = (1 - u^2/E^2)\sin^2(cu)$. It is nondecreasing: half its derivative equals

$$\sin(cu)\left[c(1 - u^2/E^2)\cos(cu) - \frac{u}{E^2}\sin(cu)\right] \geq 0.$$

Indeed $c > 9/10$, $1 - u^2/E^2 \geq 47/49$, $\cos(cu) \geq \cos\sqrt{2} \geq 7/45$, $0 \leq \sin(cu) \leq 1$ and $u/E^2 < 3/98$. The cosine bound is its alternating Taylor lower bound through degree six. The sine upper bound through degree five gives $\sin\sqrt{2} \leq (7/10)\sqrt{2}$. Consequently

$$M(u) \leq c^2 \sin^2(c\sqrt{2}) \leq \frac{49}{50}c^2.$$

Using $u = \sqrt{E}x$, (4) and (3), we obtain

$$(E - x^2)y(x)^2 \leq \frac{49}{50}A^2 \leq \frac{99}{100}EC_n^2 \quad (0 \leq x \leq x_0). \tag{5}$$

The lower sine comparison and (1) give

$$B(0) - B(x_0) \geq \frac{2A^2}{E^2}\int_0^{\sqrt{2}} u \sin^2 u\, du > \frac{A^2}{2E^2}.$$

For the strict last inequality, restrict the integral to $[1, \sqrt{2}]$, where $\sin u \geq \sin 1 \geq 5/6$; its lower bound is $25/72 > 1/4$. Thus (3) implies $B(x_0) < EC_n^2$. Monotonicity of B proves the bulk bound for $x_0 \leq x < \sqrt{E}$, completing all odd degrees $n \geq 3$.

For $n = 1$, put $v = x^2 \in [0, 3]$. Its squared bulk bound is $\pi v(1 - v/3) \leq e^v$. Use $\pi < 22/7$, $e^v \geq 1 + v + v^2/2$ and the positive quadratic

$$1 + v + v^2/2 - (22/7)v(1 - v/3) = 1 - 15v/7 + 65v^2/42 > 0.$$

We have therefore proved $|y(x)| \leq C_n g_n(x)$ throughout $0 \leq x < \sqrt{E}$ for every $n \geq 1$.

The clamped tail

Undo the harmless sign change. For $n \geq 3$, we prove $|y(x)| \leq |y(a_n)|$ for $x \geq a_n$. All roots r_i of $p = H_n$ are real, counted with multiplicity. For completeness, if $v_k = H_k(z)$ and $J_k = \operatorname{Im}(v_{k+1}\overline{v_k})$, the physical three-term recurrence gives

$$J_0 = 2\operatorname{Im} z, \qquad J_{k+1} = 2\operatorname{Im} z\, |v_{k+1}|^2 + 2(k+1)J_k.$$

For $\operatorname{Im} z \neq 0$, induction gives $(\operatorname{Im} z)J_k > 0$ and rules out $v_{k+1} = 0$.

Suppose $p(x) \neq 0$ and $y'(x) = 0$, and put $t = p'/p$, $q = E - x^2$. The Riccati identity $t' = 2xt - 2n - t^2$, differentiated twice, gives at this point

$$t = x, \quad t' = 1 - q, \quad t'' = 2x, \quad t''' = 2 - 2q^2.$$

mathdb:2610.00789v1 28 Oct 2026

Since $t = \sum_i (x - r_i)^{-1}$ locally away from the roots, writing $b_i = (x - r_i)^{-1}$ yields

$$\sum_i b_i^2 = q - 1, \qquad \sum_i b_i^3 = x, \qquad \sum_i b_i^4 = (q^2 - 1)/3.$$

Cauchy–Schwarz applied to (b_i) and (b_i^2) gives

$$3E \le F(q), \qquad F(q) = q^3 - q^2 + 2q + 1. \tag{6}$$

This uses neither simple roots nor a location beyond all the roots.

The polynomial F is increasing on $\mathbb{R}$, because $F'(q) = 3(q - 1/3)^2 + 5/3 > 0$. Let $b = 22/7$ and $R(E) = b - b^2/(4E)$. For $E \ge 7$, $\pi - \pi^2/(4E) \le R(E)$, since their difference is $(b-\pi)(4E-b-\pi)/(4E) \ge 0$. The exact identity

$$\begin{aligned}(3E - F(R(E)))117649E^3 &= 352947(E-7)^4 + 6535179(E-7)^3 + 40836159(E-7)^2 \\ &\quad + 89231464(E-7) + 19560668 > 0\end{aligned}$$

contradicts (6) if $x \ge a_n$, since $q \le E - a_n^2 = \pi - \pi^2/(4E)$. Thus no nonzero stationary value lies at or beyond a_n.

Now $y(x)^2 \to 0$ as $x \to \infty$. If $y(x)^2 > y(a_n)^2$ for some $x > a_n$, choose $R > x$ with $y(R)^2 < y(x)^2$. A maximum of y^2 on $[a_n, R]$ is positive and interior. Fermat's theorem makes $y' = 0$ there, a contradiction. The bulk estimate at a_n supplies the claimed tail.

The remaining degrees have elementary global square bounds

$$(2xe^{-x^2/2})^2 \le 3/2, \qquad ((4x^2 - 2)e^{-x^2/2})^2 \le 25/4.$$

The first follows from $ve^{-v} \le e^{-1}$ and $e \ge 8/3$. For the second, calculus gives the possible absolute maxima 2 and $8e^{-5/4}$; the Taylor bound $e^{5/2} \ge 4169/384 > 256/25$ suffices. These fit the source constants: for $Q_n = 1 - a_n^2/(2n+1)$, $Q_1 = \pi/3 - \pi^2/36 \le \pi/4$ and $C_1^2\pi = 4$, so $(3/2)Q_1 < C_1^2$ using $3 < \pi < 22/7$. Similarly $Q_2 \le \pi/5$, $C_2 = 2$ and $\pi < 16/5$ give $(25/4)Q_2 < C_2^2$.

Combining the bulk and tail estimates and multiplying by $e^{-x^2/2}$ proves Theorem 1.

Attribution and formal certificate

The source question and normalization are due to Guo and Liu. Classical Hermite identities, real-rootedness, oscillator comparison, Wallis limits, Taylor inequalities and Cauchy–Schwarz are ingredients, not separately claimed results. Foster–Krasikov [2] gives a different explicit oscillatory-region envelope. We make no assertion that every possible combination of earlier estimates has been excluded, and no worldwide-priority claim.

The Lean certificate has 162 declarations in 17 modules, checked with warnings treated as errors in Lean 4.34.1 and mathlib revision
`d13f23b723b8a846827a245b89c10fc7d3f11612`. The endpoint `source_conjecture_346243` is the literal source statement, using Rodrigues definitions and the stated Gamma coefficient and cutoff. Initial values, the ODE comparison, strict energy loss, real-root moment obstruction and clamped tail are proved. Its only axioms are `propext`, `Classical.choice` and `Quot.sound`. The endpoint has no residual odd-degree, comparison or energy-loss premise. The accompanying proof packet records independent review, pinned verification and the limits of the prior-art search.

References

[1] J. Guo and Y. Liu, *A modified Euler–Maclaurin formula in 1D and 2D with applications in statistical physics*, Communications in Theoretical Physics **73** (2021), 075002; arXiv:2004.10441v1. doi:10.1088/1572-9494/abf1dd.

[2] W. H. Foster and I. Krasikov, *Explicit bounds for Hermite polynomials in the oscillatory region*, LMS Journal of Computation and Mathematics **3** (2000), 307–314. doi:10.1112/S1461157000000310.

mathdb:2610.00789v1 28 Oct 2026