The adjacent-count inequality for the flat Bernoulli game

Research proof and Lean formalization
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Abstract

We prove the adjacent-count inequality in Hetyei's Conjecture 6.10. After factorial normalization, the counting sequence is a renewal sequence whose increments are negative from index 11 onward. Finite symmetric arrowhead matrices realize approximating moments. Their spectrum has one negative value, and positive-weight power inequalities propagate three exact coefficient checks to the entire tail. Classical finite power sums identify the limiting coefficients. The Lean proof includes the source's finite ordered-composition formula and all spectral, recurrence and limit bridges.

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1 The count formula and the target

Hetyei [1, (16), Conjecture 6.10] defines

$$K_n = \sum_{\substack{m_1+\cdots+m_k=n \\ m_i\geq 2}} \frac{n!}{m_1!\cdots m_k!}\prod_{i=1}^{k}(m_i-2)!, \qquad \kappa_{n+1} = (n+1)K_n. \tag{1}$$

The sum ranges over all ordered compositions; the empty composition gives $K_0 = 1$, consistently with the source's generating function and Proposition 6.7. The source proves the interpretation of these formulas as game-position counts. Our formalization starts from the explicit finite formula.

Theorem 1. *For every integer $n \geq 11$, $K_n < nK_{n-1}$. Equivalently, for every integer $n \geq 12$, $\kappa_n < n\kappa_{n-1}$.*

Put $u_n = K_n/n!$ and $b_n = u_n - u_{n-1}$ for $n \geq 1$. Splitting a composition at its first block gives

$$u_0 = 1, \quad u_1 = 0, \qquad u_{n+2} = \sum_{k=0}^{n} \frac{u_{n-k}}{(k+2)(k+1)}. \tag{2}$$

This also follows from the source's generating function $\sum u_n t^n = ((1-t)(1-\log(1-t)))^{-1}$. Theorem 1 is exactly $b_n < 0$ for $n \geq 11$. Exact rational arithmetic in (2) gives

$$b_3 = -\frac{1}{3}, \qquad b_{12} = -\frac{10819}{9979200} < -\frac{1}{2000}, \qquad b_{14} - b_{12} = -\frac{1707491}{1816214400} < 0. \tag{3}$$

We propagate these finite checks by a spectral argument, rather than extrapolating the observed signs.

2 Finite matrix realization

For $N \geq 1$, set $t_i = i/N$, $r_i = N^{-1/2}$, $s_i = \sqrt{1-t_i}$, $v_i = s_i r_i$ for $0 \leq i < N$, and form

$$H_N = \begin{pmatrix} 0 & v^{\mathsf{T}} \\ v & \operatorname{diag}(t_i) \end{pmatrix}, \qquad u_n^{(N)} = (H_N^n)_{00}, \quad b_n^{(N)} = u_n^{(N)} - u_{n-1}^{(N)}.$$

For $x = (a, z)$, direct expansion gives

$$x^{\mathsf{T}}(I - H_N)x = \sum_i (s_i z_i - r_i a)^2,$$
$$x^{\mathsf{T}}(I + H_N)x = \sum_i (s_i z_i + r_i a)^2 + 2\sum_i t_i z_i^2.$$

Thus every eigenvalue lies in $[-1, 1]$. The quadratic form of H_N is nonnegative on the codimension-one subspace $a = 0$.

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3 The spectral propagation argument

There is at most one negative eigenvalue, counted with multiplicity. Indeed, each negative eigenvector has nonzero center coordinate, and two orthonormal negative eigenvectors would yield a nonzero linear combination with center coordinate zero and negative quadratic form. Let w_λ be the squared center coordinates of an orthonormal eigenbasis. Then

$$\sum w_\lambda = 1, \qquad \sum w_\lambda \lambda = 0, \qquad \sum w_\lambda \lambda^2 = \frac{N+1}{2N} \geq \frac{1}{2}.$$

Consequently a negative eigenvalue exists. Write it as $-q$, where $0 < q \leq 1$. Since $(\lambda + q)(1 - \lambda) \geq 0$ at every spectral value, $\lambda^2 \leq (1-q)\lambda + q$; averaging gives $q \geq 1/2$. The center-moment expansion yields

$$b_n^{(N)} = \sum_\lambda w_\lambda (\lambda - 1)\lambda^{n-1}. \tag{4}$$

The following finite criterion is the main step.

Lemma 2. *Suppose a finite positive spectral measure has mass 1, mean 0, support in $\{-q\} \cup [0, 1]$, and $1/2 \leq q \leq 1$. Write b_n for (4). If $b_{12} \leq -c < 0$, $b_{14} \leq b_{12}$ and $b_3 \leq -1/4$, then for every integer $m \geq 0$,*

$$b_{12+2m} \leq -c\, 4^{-m}, \qquad b_{2m+1} \leq -4^{-m}.$$

Proof. For the even indices put

$$T = w_{-q}(1+q)q^{11}, \quad W_\lambda = w_\lambda(1-\lambda)\lambda^{11}, \quad S_m = \sum_{\lambda \geq 0} W_\lambda \big((\lambda/q)^2\big)^m.$$

Then $b_{12+2m} = q^{2m}(T - S_m)$. The hypotheses and $q \leq 1$ give $q^2(T - S_1) = b_{14} \leq b_{12} \leq q^2 b_{12} = q^2(T - S_0)$, hence $S_1 \geq S_0$. The elementary inequality $z^m \geq 1 + m(z-1)$ for $z \geq 0$ implies $S_m \geq S_0$. Thus $b_{12+2m} \leq -cq^{2m} \leq -c4^{-m}$. For odd indices, $d_\lambda = w_\lambda(1-\lambda)$ are nonnegative, $\sum d_\lambda = 1$, and $\sum d_\lambda \lambda^2 = -b_3 \geq 1/4$. The same power inequality, normalized by $1/4$, gives $-b_{2m+1} = \sum d_\lambda (\lambda^2)^m \geq 4^{-m}$. □

4 Exact coefficient limits and strictness

Let $A_n = u_n^{(N)}$ and let $z_{n,i}$ be the noncentral coordinates of $H_N^n e_0$. The relations

$$z_{n+1,i} = v_i A_n + t_i z_{n,i}, \qquad A_{n+1} = \sum_i v_i z_{n,i}, \qquad z_{0,i} = 0$$

give the exact convolution

$$A_{n+2} = \sum_{k=0}^{n} p_k^{(N)} A_{n-k}, \qquad p_k^{(N)} = \frac{1}{N} \sum_{i=0}^{N-1} (1 - t_i) t_i^k. \tag{5}$$

These coefficients tend to $1/((k+2)(k+1))$. One may use Riemann sums; the formal proof instead uses the classical Faulhaber identity [2]:

$$M_N(k) := \frac{1}{N} \sum_{i=0}^{N-1} (i/N)^k = \sum_{j=0}^{k} \frac{B_j \binom{k+1}{j}}{k+1} N^{-j} \longrightarrow \frac{1}{k+1}.$$

Here $B_1 = -1/2$. Since $p_k^{(N)} = M_N(k) - M_N(k+1)$, the limit follows.

Induction in (5) and (2) proves $u_n^{(N)} \to u_n$ for every fixed n. The strict inequalities in (3) imply that for all sufficiently large N,

$$b_{12}^{(N)} \le -1/2000, \qquad b_{14}^{(N)} \le b_{12}^{(N)}, \qquad b_3^{(N)} \le -1/4.$$

Lemma 2 applies with constants independent of N. Passing to the limit gives

$$b_{12+2m} \le -\frac{1}{2000} 4^{-m} < 0, \qquad b_{2m+1} \le -4^{-m} < 0.$$

Every integer $n \ge 11$ is covered. This proves Theorem 1.

5 Formal scope and research status

The development uses Lean 4.34.1 with mathlib revision `d13f23b723b8a846827a245b89c10fc7d3f11612`. It contains 117 authored declarations in 16 modules. The endpoint `FlatBernoulli.source_conjecture_319292` has only the hypothesis $n \ge 12$. The finite factorial count, first-block composition bijection, matrix sums of squares, negative-eigenvalue multiplicity argument, actual center-moment convolution, coefficient limits, exact rational checks and strict limiting bounds are all included. The original game-position type and the source's enumeration theorem are not re-formalized.

The use of finite spectral decomposition, integer-power convexity and Faulhaber's formula is classical; no new general machinery is claimed. The contribution is one resolution of the source's explicit all-order conjecture. The finite checks and supporting lemmas are ingredients of this single result. The source's ancillary printed equation(23), whose integral sign differs from the coefficient identity, is not used as a target or counted as an outcome.

The bounded prior-art investigation includes the source, relevant later papers, reciprocal-log coefficient references, OEIS and public proof repositories. It cannot certify worldwide priority. Access to the final JCTA body and one game-theory citer remains limited; no resolution is inferred from an empty citation corpus. The independent checks and replay instructions accompany the proof files.

References

[1] G. Hetyei, *Decomposition of strongly Bernoulli type games into elementary kernel factors*, arXiv:0912.0573v1 (2009), especially(16), Proposition 6.7 and Conjecture 6.10. Published in J. Combin. Theory Ser. A 117(8) (2010), 1107–1126, doi:10.1016/j.jcta.2010.01.003.

[2] The mathlib community, *Mathlib*, pinned revision `d13f23b7`; `NumberTheory/Bernoulli.lean`, theorem `sum_range_pow` (classical Faulhaber formula).

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