A continuous critical-branch obstruction to C*-algebraic Sendov conjectures

Research proof and Lean formalization
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Abstract

We give a cubic in the commutative unital C*-algebra $C([0,1],\mathbb{C})$ whose roots have norm at most $9/10$, whose derivative splits completely into two distinct continuous critical branches of norm less than one, and whose critical branches are positive continuous convex combinations of the original roots. Nevertheless, every critical element has distance greater than $21/20$ from one fixed root. This contradicts Conjecture 3 of Krishna's published C*-algebraic Sendov paper, originally Conjecture 2.5 of its preprint, and its completely split derivative variant (published Conjecture 4). The obstruction concerns the existence of one global continuous critical element. It does not contradict scalar Sendov. The two source formulations are treated as one combined outcome.

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1 Source statement and the counterexample

For a unital C*-algebra, Krishna [1, Definition 2] defines

$$\overline{\mathbb{D}^*}(a,r)=\{z\in\mathcal{A}:(z-a)(z-a)^*\le\sqrt{r}\,1\}.$$

Only $r=1$ occurs here. In $\mathcal{A}=C([0,1],\mathbb{C})$, equipped with its pointwise operations and supremum norm, this condition is precisely $|z(t)-a(t)|\le 1$ for every $t\in[0,1]$. For $p(z)=\prod_{j=0}^{2}(z-a_j)$ the source defines $p'(z)$ by the sum of the three products obtained by deleting one factor, preserving order. The algebra used here is commutative, so this is also ordinary formal polynomial differentiation.

Published Conjecture 3 [1, p. 60], corresponding to preprint Conjecture 2.5 [2] and MathDB 353909, assumes that all a_j belong to $\overline{\mathbb{D}^*}(0,1)$, that p' has two roots $b_k\in\overline{\mathbb{D}^*}(0,1)$, and that each b_k has a representation

$$b_k=\sum_{j=0}^{2}\omega_{kj}a_j,\qquad \sum_{j=0}^{2}\omega_{kj}=1,\qquad \omega_{kj}\ge 0 \text{ in } \mathcal{A}. \tag{1}$$

It asserts that for each a_j there is a zero b of p' in $\overline{\mathbb{D}^*}(a_j,1)$. Published Conjecture 4 replaces the critical-root and convexity hypotheses with complete linear factorization of the derivative.

Theorem 1. *There exist three elements $a_0,a_1,a_2\in C([0,1],\mathbb{C})$ of norm at most $9/10$ and two distinct elements B_+,B_- such that:*

1. *p is monic of degree three and $p'(z)=3(z-B_+)(z-B_-)$ for every $z\in\mathcal{A}$;*
2. *$\|B_\pm\|<9/10$, and both $B_\pm$ satisfy (1) with strictly positive continuous real-valued coefficients;*
3. *every $b\in\mathcal{A}$ satisfying $p'(b)=0$ is either B_+ or B_-, and $\|a_0-b\|>21/20$.*

Consequently the conclusions of both source formulations fail.

Put $u(t)=e^{\pi it}$ and define

$$a_0=\frac{9}{10},\qquad a_1(t)=\frac{27}{40}u(t),\qquad a_2(t)=-\frac{27}{40}u(t). \tag{2}$$

The three roots are pairwise distinct at every point: $|a_0|=9/10$, $|a_1|=|a_2|=27/40$, and u is nowhere zero. In particular all roots lie strictly inside the required unit disc.

2 Continuous critical branches and complete splitting

Use the principal complex square root to set

$$h(t)=1+\frac{16}{27}\overline{u(t)}^{\,2},\qquad q(t)=\frac{3\sqrt{3}}{4}u(t)\sqrt{h(t)},\qquad B_\pm(t)=\frac{3}{10}(1\pm q(t)). \tag{3}$$

Since $\operatorname{Re} h \geq 11/27 > 0$, these functions are continuous. Also $u\overline{u} = 1$, so direct multiplication gives

$$q^2 = 1 + \frac{27}{16}u^2. \tag{4}$$

The right-hand side has modulus at least $11/16$. Thus q is nowhere zero, and the two branches are distinct everywhere. Furthermore, $|q|^2 = |q^2| \leq 43/16 < 4$. Consequently

$$|B_\pm(t)| \leq \frac{3}{10}(1 + |q(t)|) < \frac{9}{10}.$$

Compactness makes this a strict supremum-norm bound as well.

Expanding the source's literal deleted-factor derivative yields

$$\begin{aligned} p'(z) &= (z-a_1)(z-a_2) + (z-a_0)(z-a_2) + (z-a_0)(z-a_1) \\ &= 3z^2 - \frac{9}{5}z - \frac{729}{1600}u^2 = 3(z-B_+)(z-B_-). \end{aligned}$$

The last equality follows from (4). It holds both as an identity in $\mathcal{A}[z]$ and after evaluation at every $z \in \mathcal{A}$.

Lemma 1. *The only zeros of p' in $\mathcal{A}$ are B_+ and B_-.*

Proof. If $p'(b) = 0$, then at each t the scalar factorization gives $b(t) = B_+(t)$ or $b(t) = B_-(t)$. The continuous function

$$F(t) = \frac{(10/3)b(t) - 1}{q(t)}$$

therefore takes values in $\{-1, 1\}$. The interval is connected, so F is constant. Its two possible values give the two asserted global branches. This argument classifies all continuous critical elements despite the zero divisors in the function algebra. □

3 Positive convex weights and the endpoint obstruction

Fix either branch B. As the three scalar roots are simple at every t, no critical point can equal any $a_j(t)$: evaluating p'_t at $a_j(t)$ gives the product of the other two nonzero differences. Thus

$$r_j(t) = \frac{1}{|B(t) - a_j(t)|^2}, \qquad R(t) = \sum_{j=0}^{2} r_j(t), \qquad \omega_j(t) = \frac{r_j(t)}{R(t)}$$

are continuous, r_j, R, ω_j are strictly positive, and $\sum_j \omega_j = 1$. Pointwise division of $p'(B) = 0$ by the nowhere-zero product $p(B)$ gives $\sum_j (B - a_j)^{-1} = 0$. Conjugating this equality gives $\sum_j r_j(B - a_j) = 0$, hence $B = \sum_j \omega_j a_j$. Real-valued nonnegative continuous functions are positive elements of $\mathcal{A}$, so these are exactly the coefficients required in (1).

At the endpoints put $Q = \sqrt{43}/4$. Formula (3) gives

$$q(0) = Q > \frac{3}{2}, \qquad q(1) = -Q, \qquad B_-(0) = B_+(1) = \frac{3}{10}(1 - Q).$$

Therefore

$$|a_0(0) - B_-(0)| = |a_0(1) - B_+(1)| = \frac{3}{5} + \frac{3\sqrt{43}}{40} > \frac{21}{20} > 1. \tag{5}$$

Each branch fails the unit-disc condition at one endpoint. Taking the supremum and applying Lemma 1 proves Theorem 1. The actual endpoint distance is approximately 1.0918078893.

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4 Scope and formal certificate

The construction retains complete derivative splitting, strictly interior roots and critical roots, and positive continuous convex weights. The failure is the simultaneous requirement of continuity and one uniform nearby critical element for a prescribed root. The underlying connectedness and square-root mechanism is classical and receives no separate novelty claim. The scalar pointwise Sendov statement is compatible with the example.

The commutative source formulation, MathDB 353908, also assumes density of invertibles. That additional hypothesis is not part of the formal target here; the formal source refutation is MathDB 353909 and the published completely split derivative variant, treated together. A bounded prior-art review found no previous counterexample satisfying these source hypotheses. This does not establish an absolute priority claim.

The accompanying certificate uses Lean 4.34.1 and mathlib revision `d13f23b723` (the full revision is pinned in the packet). It contains 136 declarations in eight modules. The analytic module uses the actual complex exponential and principal square root. The function algebra is the concrete unital commutative C*-algebra above, with its usual star order; the disc definition retains the source's $\sqrt{r}$. The certificate proves the cubic polynomial identity in $\mathcal{A}[z]$, not merely a pointwise numerical identity.

The topology module proves global selection rigidity on any preconnected space. The convex-weight module derives continuous strictly positive real weights from the literal deleted-product derivative, and the source bridge proves positivity in the C*-order and the algebraic sum and combination identities. Thus the source hypotheses are discharged for the displayed functions. The closed declarations `source_counterexample_353909` and `source_counterexample_published_conjecture4` negate the literal cubic specializations of the two conjectures; they retain no classification, factorization, positivity, or existence premises. The combined declaration `source_complete_counterexample` includes the degree, norm bounds, complete derivative factorization, and obstruction for every critical element. Every declaration is checked with warnings treated as errors and its axiom dependencies inspected; only `propext`, `Classical.choice`, and `Quot.sound` occur. Numerical checks are supplementary to these proofs.

References

[1] K. M. Krishna, *C*-algebraic Gauss–Lucas theorem and C*-algebraic Sendov's conjecture*, Vestnik KRAUNC. Fiz.-Mat. Nauki **54**(1) (2026), 56–63. doi:10.26117/2079-6641-2026-54-1-56-63.

[2] K. M. Krishna, same title, arXiv:2203.06916v1 (2022).

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