Diagonal concavity does not imply concavity for elementary-symmetric sums

Research proof and formalization packet
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Abstract

We refute Lachaume's diagonal-concavity conjecture with the explicit symmetric polynomial $f = 1 + 2\sigma_1 + \sigma_2 + 2\sigma_3 + \sigma_4$ in four variables. Its diagonal restriction is $P(t) = t^4 + 8t^3 + 6t^2 + 8t + 1$. The identity $4PP'' - 3(P')^2 = -144(t^2-1)^2$ proves concavity of $P^{1/4}$ on the whole positive axis, but an exact midpoint inequality shows that $f^{1/4}$ is not concave on the positive orthant. All inputs and all five coefficients are strictly positive. The complete statement, including the literal elementary symmetric functions, is certified in Lean. A related real-rootedness converse is already false by an older example and translation, and is not claimed as a new result here.

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1 The source questions and the construction

For $x = (x_1, \dots, x_n)$ write

$$\sigma_k(x) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}, \qquad \sigma_0(x) = 1.$$

Let $\Gamma_n = (0,\infty)^n$. Lachaume [1, Conjecture 1, p. 3] asks whether, for $a_0, \dots, a_p \ge 0$, concavity of $\left(\sum_{k=0}^p a_k \sigma_k\right)^{1/p}$ on Γ_n is equivalent to concavity of its restriction to the positive diagonal.

Theorem 1. *Set $n = p = 4$, $(a_0, \dots, a_4) = (1,2,1,2,1)$, and*

$$f = 1 + 2\sigma_1 + \sigma_2 + 2\sigma_3 + \sigma_4, \qquad P(t) = f(t,t,t,t) = t^4 + 8t^3 + 6t^2 + 8t + 1.$$

Then:

1. *$P^{1/4}$ is concave on $(0,\infty)$.*
2. *$f^{1/4}$ is positive and twice continuously differentiable on Γ_4, but is not concave there.*

Thus the diagonal-concavity implication in the cited conjecture fails.

2 Concavity of the diagonal

Differentiating gives

$$P' = 4t^3 + 24t^2 + 12t + 8, \qquad P'' = 12t^2 + 48t + 12.$$

Expansion yields the exact identity

$$4PP'' - 3(P')^2 = -144(t^2-1)^2. \tag{1}$$

All coefficients of P are positive, so $P(t) > 0$ for $t > 0$. For the genuine positive fourth root $h(t) = P(t)^{1/4}$,

$$h''(t) = \frac{4P(t)P''(t) - 3P'(t)^2}{16P(t)^{7/4}} \le 0 \qquad (t > 0).$$

Consequently h is concave on the full positive axis.

3 A strictly positive midpoint witness

Fix the first coordinate at 2. Direct expansion of the elementary symmetric functions gives

$$g(t) := f(2,t,t,t) = 5 + 12t + 15t^2 + 4t^3. \tag{2}$$

Consider the strictly positive points

$$A = (2, 1/20, 1/20, 1/20), \quad B = (2, 3/20, 3/20, 3/20), \quad C = \frac{A+B}{2} = (2, 1/10, 1/10, 1/10).$$

Equation (2) gives

$$f(A) = \frac{2819}{500}, \qquad f(C) = \frac{3177}{500}, \qquad f(B) = \frac{7151}{1000}.$$

The following are exact rational inequalities:

$$\left(\frac{15409}{10000}\right)^4 < \frac{2819}{500}, \qquad \frac{3177}{500} < \left(\frac{15877}{10000}\right)^4, \qquad \left(\frac{16352}{10000}\right)^4 < \frac{7151}{1000}.$$

Monotonicity of the fourth power on nonnegative numbers now proves

$$f(C)^{1/4} < \frac{15877}{10000} < \frac{1}{2}\left(\frac{15409}{10000} + \frac{16352}{10000}\right) < \frac{f(A)^{1/4} + f(B)^{1/4}}{2}.$$

The middle rational gap is $7/20000$. This strictly reverses the midpoint inequality required by concavity. Moreover $f > 0$ on Γ_4 because all its coefficients and coordinates are positive; hence its fourth root is smooth there. This completes the proof of Theorem 1.

4 Formal certificate and scope

The accompanying three Lean modules contain 61 declarations, replayed from fresh source copies with Lean 4.34.1 and mathlib revision `d13f23b7`. The verifier treats warnings as errors and checks the axioms of every declaration; only `propext`, `Classical.choice`, and `Quot.sound` occur. The source-conjecture endpoint includes strict positivity and C^2 regularity, universal diagonal concavity, and failure of concavity on the positive orthant. An ancillary endpoint checks a nonreal root and the actual polynomial derivatives; no new real-rootedness-converse claim is attached to that endpoint. The elementary symmetric functions are literal subset sums, identified with Mathlib's `MvPolynomial.esymm`. No numerical approximation is used in the certificate.

The distinct real-rooted-diagonal sufficient condition is not refuted. Lachaume's later work [2, Theorem 2.7(d)] establishes that condition using Walsh's theorem. Carptopus [3] identifies the derivative convolution in the original Conjectures 4 and 5 with a rescaled Walsh convolution and explicitly excludes Conjectures 1 and 3 from its scope. We claim neither a new Walsh theorem nor a resolution of those neighboring questions. The discovery claim here concerns only the diagonal-concavity implication.

An important earlier example is Zaimi's $Q(x) = x^4 - 1$ [4], which satisfies $4QQ'' - 3(Q')^2 = -48x^2$ and has nonreal roots. Translation preserves this differential inequality. Thus $Q(x+2) = x^4 + 8x^3 + 24x^2 + 32x + 15$ already gives a positive-coefficient counterexample to the separate real-rootedness converse in [1, Conjecture 3]. That failure is prior work, not a new contribution. Our quartic also has a nonreal root, but the result proved here is the strictly positive midpoint failure despite universal diagonal concavity. The closest citing chapter [5, Section 5, pp. 166–183] is compared in the packet: it treats the known quadratic and adjacent-term cases, then powers, products and ratios of elementary symmetric functions; no resolution of the diagonal-concavity question was found there. The relevant product-minus-power construction in [5, Theorem 4.17] also explains concavity of the shifted Zaimi polarization. The inspected sources and access limits are recorded in the packet. No worldwide priority claim or complete literature census is asserted.

References

[1] X. Lachaume, *On the concavity of a sum of elementary symmetric polynomials*, arXiv:1712.10327v1 (2017), https://arxiv.org/abs/1712.10327.

[2] X. Lachaume, *The constraint equations of Lovelock gravity theories: a new σ_k-Yamabe problem*, J. Math. Phys. **59**, 072501 (2018), arXiv:1712.04528v2, doi:10.1063/1.5023758.

[3] Carptopus, *A Walsh-convolution proof of two conjectures of Lachaume*, v0.1-beta (4 September 2026), doi:10.5281/zenodo.22299983.

[4] G. Zaimi, Answer to MathOverflow question 248816 (2016), https://mathoverflow.net/a/248836.

[5] D. Andrica, S. Rădulescu, and M. Rădulescu, *Some New Methods for Generating Convex Functions*, in *Differential and Integral Inequalities*, Springer (2019), pp. 135–229, doi:10.1007/978-3-030-27407-8_4.

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