A cubic counterexample to three C*-algebraic polynomial inequalities

Research proof and formalization packet
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Abstract

We give one degree-three matrix construction refuting Krishna's C*-algebraic Schoenberg, de Bruin–Sharma and Kushel–Tyaglov conjectures. Its ordered deletion derivative factors as required for every argument in $M_2(\mathbb{C})$, including arguments that do not commute with its roots. Both adjoint versions of each proposed inequality fail, with exact negative quadratic forms. A complete 36-declaration Lean certificate proves the factorization and all six Hermitian positivity failures. The classical scalar theorems are unaffected.

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1 The source assertions

Krishna [1], Section 2, considers $P(z) = (z - a_1)\cdots(z - a_d)$ over a C*-algebra $\mathcal{A}$ and defines

$$P'(z) = \sum_{j=1}^{d} (z - a_1)\cdots\widehat{(z - a_j)}\cdots(z - a_d), \qquad z \in \mathcal{A}.$$

The factors retain their order. The hypotheses of his Conjectures 2.1, 2.3 and 2.4 require a factorization $P'(z) = d(z - b_1)\cdots(z - b_{d-1})$ on the whole algebra. They impose no normality or self-adjointness condition on the a_j. MathDB 355008 and 355009 record the first two conjectures.

For centered roots $\sum a_j = 0$ and degree $d = 3$, write

$$S = \sum_j a_j a_j^*, \quad T = \sum_j (a_j a_j^*)^2, \quad C = \sum_j a_j^2, \quad E = \sum_j a_j^2 (a_j^2)^*.$$

The left-adjoint versions of the three assertions specialize to

$$\sum_{k=1}^{2} b_k b_k^* \preceq S/3, \tag{1}$$

$$\sum_{k=1}^{2} (b_k b_k^*)^2 \preceq 2S^2/9 - T/3, \tag{2}$$

$$\sum_{k=1}^{2} (b_k b_k^*)^2 \preceq -T + S^2/9 + CC^*/9 + 2E/3. \tag{3}$$

Each has a corresponding adjoint-reversed assertion: replace $a_j a_j^*$ and $b_k b_k^*$ by $a_j^* a_j$ and $b_k^* b_k$, CC^* by C^*C, and E by $\sum_j (a_j^2)^* a_j^2$. In the third assertion the terms involving $\sum a_j$ vanish. We take $n = d = 3$, so the source's mixed n, d indexing in Conjecture 2.4 has no bearing on this example.

The scalar predecessors are known theorems; see Cheung–Ng [2] and Kushel–Tyaglov [3]. The result below concerns their proposed C*-algebraic extensions.

2 One matrix construction

Theorem 1. *All three C*-algebraic conjectures fail at degree 3 in $\mathcal{A} = M_2(\mathbb{C})$. One centered cubic, with a derivative factorization valid for every $z \in \mathcal{A}$, violates both adjoint versions of all three inequalities.*

Proof. Put

$$R = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \qquad (a_1, a_2, a_3) = (R, -2R, R), \qquad (b_1, b_2) = (I_2, -I_2).$$

Then $R^2 = I_2$ and $\sum a_j = 0$. Without interchanging z and R, the ordered derivative expands to

$$\begin{aligned} P'(z) &= (z+2R)(z-R) + (z-R)^2 + (z-R)(z+2R) \\ &= 3z^2 - 3R^2 = 3(z-I_2)(z+I_2). \end{aligned}$$

Thus the full factorization hypothesis holds, even at noncommuting arguments. Each a_j is a zero of the displayed cubic and each b_k is a zero of its deletion derivative.

Let

$$H = RR^* = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}, \qquad J = R^*R = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

We have $S = 6H$, $T = 18H^2$, $C = 6I_2$ and $E = 18I_2$. Both critical-point sums in (1)–(3) equal $2I_2$. Consequently, the right sides of those inequalities are respectively

$$2H, \qquad 2H^2, \qquad 16I_2 - 14H^2.$$

The reversed versions have the same expressions with H replaced by J.

For $v = (1,2)^T$ and $w = (2,-1)^T$, $Hv = (1,0)^T$, $Jw = (0,-1)^T$ and $v^*v = w^*w = 5$. It follows that

$$v^*2(H - I_2)v = v^*2(H^2 - I_2)v = -8, \qquad w^*2(J - I_2)w = w^*2(J^2 - I_2)w = -8.$$

These refute the first two pairs of inequalities. Finally, $(H^2)_{11} = (J^2)_{22} = 29$, so

$$e_1^*\big(14I_2 - 14H^2\big)e_1 = e_2^*\big(14I_2 - 14J^2\big)e_2 = -392.$$

These are the defects in the third pair. All six defects are real symmetric, hence Hermitian, and none is positive semidefinite. □

The roots even commute pairwise. Their nonnormality is essential to this construction: similarity that is not unitary does not preserve the adjoint/order expressions appearing in these conjectures. No new theorem about nonnormal involutions is being asserted.

3 Formal scope and provenance

The two Lean modules construct the literal ordered cubic and deletion derivative, prove its factorization for every complex 2×2 matrix, and prove centeredness, the actual root/critical-point evaluations, all six exact defect matrices, Hermitianity and failure of positive semidefiniteness. The final closed theorem is `CstarPolynomial.source_all_three_conjectures_false`. In particular, the full terms of Conjecture 2.4 are formalized before their centered specialization; they are not assumed to vanish.

Strict replay uses Lean 4.34.1 and mathlib revision `d13f23b723b8a846827a245b89c10fc7d3f11612`, with exact dependency pins included. Every declaration type and its axioms are recorded; only `propext`, `Classical.choice` and `Quot.sound` occur. A separate rational-arithmetic script checks the six expressions, while Lean supplies the universal proof. Acceptance and the status of independent review and bounded literature searches are recorded separately. No absolute priority claim is made. The construction and its related failures constitute one outcome.

References

[1] K. Mahesh Krishna, *C*-algebraic Schoenberg Conjecture*, arXiv:2206.06653v1 (2022), Conjectures 2.1, 2.3 and 2.4.

[2] W. S. Cheung and T. W. Ng, *A companion matrix approach to the study of zeros and critical points of a polynomial*, J. Math. Anal. Appl. **319** (2006), 690–707.

[3] O. Kushel and M. Tyaglov, *Circulants and critical points of polynomials*, J. Math. Anal. Appl. **439** (2016), 634–650.

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