mathdb:2610.00773v1 28 Oct 2026

# A Hermitian counterexample to C*-algebraic Novak positivity

Research proof and formalization packet

October 26, 2026

**Abstract**

Krishna's C*-algebraic Novak conjecture proposes a positive matrix built from operator cosines of differences of self-adjoint inputs. We give a counterexample in $M_2(\mathbb{C})$ with four inputs and two coordinates. Every entry of the resulting kernel is a real scalar multiple of the identity, so the matrix is Hermitian. An explicit vector has quadratic form $\cos(\pi\sqrt{15})-3 \le -2$. Thus the failure is independent of product ordering and already occurs before the proposed constant lower bound is subtracted. A 36-declaration Lean certificate constructs the literal exponential-defined source matrix and proves its failure of positive semidefiniteness.

## 1 The source conjecture

For an element $T$ of a unital C*-algebra, define

$$\cos T = \frac{\exp(iT) + \exp(-iT)}{2}.$$

Krishna [1], Definition 6 and Conjecture 4.2, asks whether

$$\left[\prod_{\ell=1}^{d} \frac{1+\cos(X_{j,\ell} - X_{k,\ell})}{2} - \frac{1}{n}\right]_{j,k=1}^{n} \succeq 0 \tag{1}$$

for all $n, d \ge 2$ and all self-adjoint inputs $X_{j,\ell}$. Here scalar constants denote multiples of the algebra's identity. The conjecture is numbered 4.3 in the 2021 preprint; the published Theorem 4.3 proves the commutative case. MathDB 351633 records the unrestricted conjecture.

**Theorem 1.** *The assertion (1) is false for $\mathcal{A} = M_2(\mathbb{C})$, $n = 4$ and $d = 2$. It fails for a Hermitian matrix all of whose entries belong to the center of $\mathcal{A}$.*

The original scalar Novak theorem [2] and Krishna's commutative theorem are unaffected. We use only standard finite-dimensional matrix calculus; no new general theorem about positive definite radial functions is claimed.

## 2 The four inputs

Let

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy $Z^2 = S^2 = I_2$ and $ZS + SZ = 0$. Set

$$U = 2\pi Z, \qquad V = \frac{\pi}{4}(-7Z + \sqrt{15}\,S), \qquad (P_1, P_2, P_3, P_4) = (0, U, U+V, V). \tag{2}$$

Take $X_{j,1} = P_j$ and $X_{j,2} = 0$. All eight inputs are self-adjoint. The anticommutation identity gives

$$U^2 = V^2 = 4\pi^2 I_2, \qquad (U+V)^2 = \pi^2 I_2, \qquad (U-V)^2 = 15\pi^2 I_2. \tag{3}$$

For example, the $Z$ and $S$ coefficients of $U+V$ are $\pi/4$ and $\pi\sqrt{15}/4$, whose squared sum is $\pi^2$.

**Lemma 2.** *If $M^2 = r^2 I$ and $r > 0$, then its exponential-defined cosine is $\cos M = (\cos r) I$.*

*Proof.* Put $E_+ = rI + M$ and $E_- = rI - M$. Then $E_+ M = rE_+$ and $E_- M = -rE_-$. Applying the exponential power series shows

$$E_+ \cos M = (\cos r) E_+, \qquad E_- \cos M = (\cos(-r)) E_- = (\cos r) E_-.$$

Add the identities and use $E_+ + E_- = 2rI$ to cancel the nonzero scalar $2r$. This argument does not require diagonalization. □

By (3), Lemma 2, and the evenness of cosine,

$$\cos U = \cos V = I_2, \qquad \cos(U+V) = -I_2, \qquad \cos(U-V) = cI_2, \quad c = \cos(\pi\sqrt{15}) \le 1.$$

Thus, with $t = (1+c)/2$, the uncentered source kernel is

$$\left[\prod_{\ell=1}^{2} \frac{1+\cos(X_{j,\ell} - X_{k,\ell})}{2}\right]_{j,k=1}^{4} = K \otimes I_2, \qquad K = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & t \\ 0 & 1 & 1 & 1 \\ 1 & t & 1 & 1 \end{pmatrix}. \tag{4}$$

Indeed, the second-coordinate factor is always $I_2$. All factors appearing here are scalar matrices, so either ordered or symmetrized multiplication gives exactly (4).

*Proof of Theorem 1.* Let $y = (1, -1, 1, -1)^T$. Since $\sum_j y_j = 0$, the constant matrix $J_4$ satisfies $y^T J_4 y = 0$. Direct multiplication yields

$$y^T (K - J_4/4) y = y^T K y = -4 + 2t = c - 3 \le -2 < 0.$$

Consequently the literal matrix in (1), which is $(K - J_4/4) \otimes I_2$, has a negative quadratic form on $y \otimes (1,0)^T$. It is Hermitian but not positive semidefinite. The same computation also rules out positivity of the uncentered kernel. □

## 3 Formal scope and provenance

The three-module Lean certificate has 36 declarations. It constructs the operator cosine from `NormedSpace.exp`, proves all input and cosine identities, and defines the literal source matrix. Its endpoint, `Novak.source_counterexample_351633`, proves self-adjoint inputs, a Hermitian source matrix, and failure of positive semidefiniteness, without residual hypotheses. Every declaration passes an isolated strict replay with Lean 4.34.1 and pinned mathlib revision `d13f23b723b8a846827a245b89c10fc7d3f11612`; only `propext`, `Classical.choice` and `Quot.sound` occur as logical axioms.

The published source statement and cosine definition were checked directly on page 799. Source and literature screening, independent review and artifact receipts are recorded separately in the accompanying research packet. No absolute priority or separate discovery credit for helpers is claimed.

## References


[1] K. M. Krishna, C*-algebraic Schur product theorem, Pólya–Szegő–Rudin question and Novak's conjecture, *J. Korean Math. Soc.* **59** (2022), no. 4, 789–804. doi:10.4134/JKMS.j210627. Preprint: arXiv:2108.06662.

[2] J. Vybíral, A variant of Schur's product theorem and its applications, *Adv. Math.* **368** (2020), 107140. doi:10.1016/j.aim.2020.107140.


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