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A strict valley in Keith's multicolor partition array

Research proof and formalization packet

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Abstract

Keith conjectured that every linear cut through a box-partition coefficient array obtained by fixing one color variable to one is unimodal. We give a three-color counterexample in the 25×25 box at partition size 325. Three consecutive points on a line of step $(1, -2)$ have coefficients 53130, 50600, and at least 52900, respectively. An injection from three-element subsets of the allowed part sizes supplies the final bound. A Lean certificate verifies the literal box-partition polynomial and its failure of unimodality. The original two-color conjecture of Dousse and Kim is not refuted.

1 The source question

A partition fits in an $M \times N$ box if it has at most N parts and every part is at most M. In Keith's $(k, 1)$-coloring convention, each distinct part size receives one of k colors; a variable records how many distinct sizes receive that color. Let a_r count ordinary partitions of a prescribed integer n in the box having exactly r distinct sizes. For three colors, setting the third color variable to one gives

$$F(x, y) = \sum_{r=0}^{\min(M,N)} a_r (1 + x + y)^r. \tag{1}$$

This is both the direct counting rule and the specialization of Keith's Theorem 3 [1].

Conjecture 1 of [1] says every linear cut through the remaining coefficient array is unimodal. The discussion on printed page 8 explicitly includes slant cuts with step $(1, -2)$. Thus the line used below is within the stated intended scope. MathDB 345259 records this multicolor question. It extends Dousse–Kim's two-color conjecture [2, Conjecture 7.1(ii)]; we make no assertion against that original one-variable conjecture.

Theorem 1. *Take $M = N = 25$ and $n = 325$, and define $C(i, j) = [x^i y^j]F(x, y)$ by (1). Then*

$$C(20, 5) = 53130, \qquad C(21, 3) = 50600, \qquad C(22, 1) \geq 52900.$$

Consequently the full nonnegative-lattice cut $y = 45 - 2x$ is not unimodal. In particular, Keith's three-color linear-cut conjecture is false.

Here unimodal means weakly increasing up to a peak and weakly decreasing thereafter. A strict valley at three consecutive points rules this out, irrespective of the values elsewhere on the cut. All three coefficients in Theorem 1 are positive.

2 Partitions near the maximum number of distinct sizes

Throughout the proof the allowed sizes are $1, \dots, 25$ and their sum is 325. The staircase partition containing each size once is therefore the unique partition with 25 distinct sizes. Thus $a_{25} = 1$.

There is no partition with 24 distinct sizes. Indeed, let b be the omitted size. The distinct sizes sum to $325 - b$, and the restriction to at most 25 parts permits at most one additional part. That part would have to have size b, contradicting its omission. Hence $a_{24} = 0$.

Lemma 2. *There are at least* $\binom{25}{3} = 2300$ *partitions of* 325 *in the box with exactly* 23 *distinct sizes.*

Proof. Choose $1 \le u < v < w \le 25$, and put $d = u + w - v$. Then $u < d < w$. Starting with the staircase, perform the following replacement:

$$\begin{cases} \text{remove } u, w \text{ and add } v, d, & v \le d, \\ \text{remove } d, v \text{ and add } u, w, & d < v. \end{cases}$$

The sum and number of parts are unchanged, and the added sizes are disjoint from the removed sizes. The result has 25 parts and exactly 23 distinct sizes. When $v = d$, the first case adds two further copies of that size and still has exactly the claimed support.

The partition determines its two missing sizes and its multiset of two extra parts, obtained by removing one occurrence of every present size. If the smaller missing size is below the smaller extra size, the first case applies: the missing sizes are u, w, and the smaller extra size is v. Otherwise the second case applies: the extra sizes are u, w and the larger missing size is v. The original triple is therefore uniquely recovered. This proves injectivity and the lemma. □

3 The coefficient valley

For a partition having r distinct sizes, the number of color assignments contributing to $x^i y^j$ after the third variable is set to one is

$$\binom{r}{i}\binom{r-i}{j}. \tag{2}$$

Choose i sizes for the first color, j of the remaining sizes for the second, and give the rest the third color. If $r < i + j$ the weight is zero.

At $(20, 5)$ only $r = 25$ contributes, giving $\binom{25}{5} = 53130$. At $(21, 3)$ only $r = 24, 25$ could contribute; the former is absent, and the staircase gives

$$\binom{25}{21}\binom{4}{3} = 50600.$$

At $(22, 1)$ each of the 2300 distinct partitions from Lemma 2 contributes $\binom{23}{22}\binom{1}{1} = 23$. All other contributions are nonnegative, so

$$C(22, 1) \ge 23\binom{25}{3} = 52900 > 50600.$$

This proves Theorem 1. The three exponent pairs are consecutive points on the indicated line. The full cut is indexed by $x = 0, \dots, 22$, so the nonunimodality is not produced by an arbitrary reordering or omission of lattice points.

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4 Formal verification and attribution

The certificate represents ordinary partitions with mathlib's `Nat.Partition`: a multiset of positive natural numbers with specified sum. It filters by the literal box restrictions, constructs the polynomial (1) as a sum over those partitions, and proves its coefficient formula (2) using polynomial binomial identities. No coefficient values or counting identities are endpoint hypotheses.

For the injection, the kernel checks all 2300 triples, their actual partition multisets, box membership, support size and a decoding left inverse. This finite certificate uses kernel reduction, not `native_decide`. The endpoint `source_full_counterexample_345259` states nonunimodality of the literal polynomial cut. The six-module certificate has 37 declarations; the only permitted axioms are `propext`, `Classical.choice` and `Quot.sound`. The separate proof packet records pinned compiler/dependency revisions and replay receipts.

The source's counting formula, the two-color predecessor, binomial coefficient identities and elementary finite-set counting are credited; they are not separate discoveries. This is one proposed source-question resolution. Initial searches found no earlier resolution of the multicolor statement, but a finite search does not certify worldwide priority. The actual arXiv source and its explicit slant-cut example were inspected. The final publisher and institutional PDF requests returned HTTP 403, so the final typeset body remains an explicit access limit. Independent review and further bounded literature checks remain required acceptance gates, recorded separately from this manuscript.

References

[1] W. J. Keith, *Restricted k-color partitions, II*, International Journal of Number Theory **17** (2021), no. 3, 591–601. doi:10.1142/S1793042120400151. arXiv:2001.08351v1, Theorems 3–4, Conjecture 1 and printed page 8.

[2] J. Dousse and B. Kim, *An overpartition analogue of q-binomial coefficients, II: Combinatorial proofs and (q,t)-log concavity*, Journal of Combinatorial Theory, Series A **158** (2018), 228–253. doi:10.1016/j.jcta.2018.03.011. arXiv:1701.07915, Conjecture 7.1.

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