mathdb:2610.00771v1 28 Oct 2026

Classical-mean power sums: the exact real-exponent ranges

Research proof and formalization packet

October 26, 2026

Abstract

For the arithmetic, geometric, harmonic and quadratic means of two positive real numbers, we determine every real exponent for which $A^n + G^n \le Q^n + H^n$ holds universally. The range is exactly $n \le 0$ or $n \ge 1$. The reverse inequality holds universally only at $n = 0$. Consequently both strict signs occur for each $0 < n < 1$. The first classification contradicts the proposed range in Meštrović–Andjić's Conjecture 1.5 in its published version. Classical two-coordinate convexity proves sufficiency; a negative second derivative at equal inputs proves necessity. Explicit positive inputs establish failure of the reverse at every nonzero exponent. A complete five-module Lean certificate verifies both classifications. The result is one combined source-question resolution; known integer cases and classical convexity tools are credited.

1 The source question and exact result

For positive real numbers x, y, write

$$H = \frac{2xy}{x+y}, \qquad G = \sqrt{xy}, \qquad A = \frac{x+y}{2}, \qquad Q = \sqrt{\frac{x^2+y^2}{2}}.$$

Meštrović and Andjić [1] prove $A^n + G^n \le Q^n + H^n$ for integer n. Their published Conjecture 1.5 proposes validity for every negative real exponent and every $n \ge 1/2$, with neither direction universally valid for $0 < n < 1/2$. The preprint numbers the same assertion as Conjecture 1.6. This is the source question represented by MathDB 338823.

Theorem 1. *For a real number n, the following equivalences hold:*

$$[\, A^n + G^n \le Q^n + H^n \text{ for all } x, y > 0 \,] \iff n \le 0 \text{ or } n \ge 1, \tag{1}$$

$$[\, Q^n + H^n \le A^n + G^n \text{ for all } x, y > 0 \,] \iff n = 0. \tag{2}$$

For every $0 < n < 1$, there are positive inputs for which each strict direction holds. All powers are ordinary real powers of positive bases.

The result addresses an entire interval of the proposed source range: the first direction fails universally for every $1/2 \le n < 1$. The source's integer results and the equality at zero are known cases, not additional claimed outcomes. Both equivalences and all witnesses constitute one classification.

mathdb:2610.00771v1 28 Oct 2026

2 Known mean comparisons and convexity

We first record elementary relations, including the source's sum and product comparisons:

$$H \le G \le A \le Q, \qquad A+G \le Q+H, \qquad QH \le AG. \tag{3}$$

For completeness, put $c = A/G \ge 1$. The identities $AH = G^2$ and $Q^2 = 2A^2 - G^2$ give

$$(H, G, A, Q)/G = (c^{-1}, 1, c, \sqrt{2c^2-1}). \tag{4}$$

Let $q = \sqrt{2c^2-1}$. The order relations are immediate. The remaining two comparisons follow from

$$q^2 - (c+1-c^{-1})^2 = \frac{(c-1)^3(c+1)}{c^2} \ge 0, \qquad c^4 - q^2 = (c^2-1)^2 \ge 0.$$

All compared quantities are positive, so these squared comparisons give $q \ge c+1-c^{-1}$ and $q/c \le c$. This proves (3). These source comparisons are credited as known; the certificate also proves them directly from the definitions.

Lemma 2. *Suppose $d \le b \le a \le c$ and $a+b \le c+d$ in an interval. For every increasing convex function f on that interval,*

$$f(a) + f(b) \le f(c) + f(d).$$

Proof. Put $c' = a+b-d$. Then $d \le b \le a \le c' \le c$ and $a+b = c'+d$. If $c' = d$, the equal-sum comparison is equality. Otherwise, applying the chord inequality on $[d, c']$ to a and b and adding gives $f(a)+f(b) \le f(c')+f(d)$. Monotonicity gives $f(c') \le f(c)$, proving the assertion. □

This is the classical two-coordinate weak-majorization argument, included to make the proof self-contained; it is not a new convexity principle.

For $n \ge 1$, apply Lemma 2 to $f(t) = t^n$ and $(a, b, c, d) = (A, G, Q, H)$, using (3). This proves the first inequality throughout that range.

For $n \le 0$, the four numbers

$$d = n \log Q \le b = n \log A \le a = n \log G \le c = n \log H$$

have $a+b \le c+d$. Indeed $AG \ge QH$, and multiplication of their logarithmic comparison by $n \le 0$ reverses its direction. Apply the lemma to the increasing convex function exp and reorder the terms. This proves the first inequality for every $n \le 0$, including equality at zero.

3 Failure for every intermediate exponent

Every $c \ge 1$ in (4) is realized by the actual positive pair

$$x_c = c + \sqrt{c^2-1}, \qquad y_c = c - \sqrt{c^2-1}. \tag{5}$$

Their product is one, their sum is $2c$, and $\sqrt{c^2-1} < c$, so both inputs are strictly positive. Define the normalized defect

$$D_n(c) = (2c^2-1)^{n/2} + c^{-n} - c^n - 1. \tag{6}$$

It is precisely $Q^n + H^n - A^n - G^n$ on the pair (5). Real powers are smooth in a neighborhood of $c = 1$, since both bases are positive there. Direct differentiation gives

$$\begin{aligned} &D_n(1) = 0, \\ &D_n'(c) = 2nc(2c^2-1)^{n/2-1} - nc^{-n-1} - nc^{n-1}, \\ &D_n'(1) = 0, \qquad D_n''(1) = 4n(n-1). \end{aligned}$$

For every $0 < n < 1$, the last value is negative. Since $D_n'(1) = 0$, differentiability of D_n' at one implies $D_n'(c) < 0$ for all $c > 1$ sufficiently close to one. The mean value theorem therefore gives $D_n(c) < D_n(1) = 0$ on such inputs. Thus every exponent in $(0, 1)$ fails the forward universal inequality. Together with sufficiency this proves (1).

An exact counterexample to the source's proposed half-power case is

$$n = \tfrac{1}{2}, \qquad x = 4 + \sqrt{15}, \qquad y = 4 - \sqrt{15}.$$

Here $(A, G, H, Q) = (4, 1, 1/4, \sqrt{31})$, and the proposed inequality would read $3 \le 31^{1/4} + 1/2$. In fact $31 < (5/2)^4$, so its right-hand side is strictly smaller than three. This witness is one instance of the same classification, not another outcome.

4 The reverse universal range

We provide strict witnesses against the reverse for every $n \neq 0$. First suppose $n > 0$. Put

$$\varepsilon = (5/4)^n - 1 > 0, \qquad c = \max\{2, (2/\varepsilon)^{1/n}\}.$$

Then $\varepsilon c^n \ge 2$. For $c \ge 2$, $q = \sqrt{2c^2 - 1} \ge (5/4)c$, since $(2c^2 - 1) - (25/16)c^2 = (7/16)c^2 - 1 > 0$. Consequently

$$Q^n = q^n \ge (5/4)^n c^n = (1 + \varepsilon)c^n \ge c^n + 2 > A^n + G^n$$

on the positive pair (5). Since $H^n > 0$, the forward inequality is strict there and the reverse fails.

If $n < 0$, instead put $c = \max\{2, 3^{1/(-n)}\}$. Then $H^n = c^{-n} \ge 3$, while $A^n = c^n \le 1$ and $G^n = 1$. Again $Q^n + H^n > A^n + G^n$. At $n = 0$, both sums are two. This proves (2). Combining these strict witnesses with the preceding negative-defect witnesses proves the assertion about both signs throughout $(0, 1)$.

mathdb:2610.00771v1 28 Oct 2026

5 Formal coverage and prior-art limits

The five modules `MeanAlgebra`, `MeanSufficiency`, `MeanNecessity`, `MeanClassification`, and `MeanConverse` contain 34 local declarations. They begin with the literal positive-input mean definitions. The certificate proves all algebraic comparisons and normalized input identities, the two-coordinate convexity lemma, sufficiency, the derivative and neighborhood-sign argument, actual positive-input counterexamples, the exact half-power example, and the reverse-direction witnesses. The endpoint `source_full_classification`, in namespace `ClassicalMeanRange`, states both equivalences without any supplied mean, derivative, normalization or witness premise.

A fresh isolated replay compiled every declaration with warnings as errors under Lean 4.34.1 and the pinned mathlib dependencies. Compiler and mathlib commits, respectively, are

`5045d0056413266e57c625dcd7c365b10e377c52`
`d13f23b723b8a846827a245b89c10fc7d3f11612.`

All declaration types and axiom reports are retained. Only `propext`, `Classical.choice`, and `Quot.sound` occur; there are no proof holes, custom axioms, unsafe declarations or native computation assumptions. Exact dependency pins and the replay script accompany the candidate.

The actual published source retains its conjecture on page 1894. Its integer theorem and sum/product comparisons, and the classical convexity principle, receive prior credit. No separate outcome is claimed for those ingredients, zero-exponent equality, an individual witness or the reverse direction.

An independent source, mathematical and formal review, including a fresh replay of all 34 declarations, found no decisive objection. Targeted literature and posted-solution searches located no earlier full resolution. Related primary comparisons were also examined: Meštrović's later chain paper [2] concerns fixed nested means; Jameson's Theorems 1–2 [3] compare a power mean M_p with $(1-p)G+pA$ and with $G^{1-p}A^p$, rather than this four-mean power-sum family. The complete publisher HTML of the latter was read even though direct PDF retrieval returned HTTP 403.

The literature assessment is bounded. Subscription citation services were unavailable, and a broad survey was only partially inspected. Citation counts and an absence of search hits do not prove priority. No claim of exhaustive worldwide novelty is made; exact sources, retrieval limits and independent review are retained in the packet. No external posting or author contact has occurred.

References

[1] R. Meštrović and M. Andjić, *Two Curious Inequalities Involving Different Means of Two Arguments*, Resonance **28**(12) (2023), 1891–1897, doi:10.1007/s12045-023-1726-x; online January 31, 2024. Preprint arXiv:1804.00542v1, Conjecture 1.6; published Conjecture 1.5.

[2] R. Meštrović, *A chain of eight inequalities involving means of two arguments*, The Teaching of Mathematics **27**(1) (2024), 27–32, doi:10.57016/TM-XDVI2817.

[3] G.J.O. Jameson, *Some inequalities for power means; a problem from "The logarithmic mean revisited"*, American Mathematical Monthly **130**(3) (2023), 276–278, doi:10.1080/00029890.2022.2153560.

mathdb:2610.00771v1 28 Oct 2026