mathdb:2610.00770v1 28 Oct 2026

Unit-circle nonvanishing for rational Hartogs-triangle polynomials

Research proof and formalization packet

October 26, 2026

Abstract

We prove the full unit-circle nonvanishing assertion of Conjecture 4.20 of Edholm and Mathew. For every coprime pair of integers $m > n \geq 1$, their degree $2(m-n)$ palindromic polynomial $Q_{m,n}$ has no root on the unit circle, and exactly $m-n$ roots in the punctured unit disc, counting multiplicities. A finite coefficient reindexing expresses its normalized boundary value as a sum of squared absolute values of floor polynomials. Two adjacent floor polynomials have a difference consisting of one monomial times $1-z$, which makes the sum strictly positive. The source coefficient normalization and the full nonvanishing theorem are verified in Lean. Earlier kernel formulas and the weighted roots-of-unity identity of Almughrabi are explicitly credited.

1 The source polynomial and the result

Fix positive integers $m > n$ with $\gcd(m,n) = 1$, and write $k = m - n$. For an integer u define the triangular coefficient

$$D_m(u) = \begin{cases} u+1, & 0 \leq u \leq m-1, \\ 2m-1-u, & m \leq u \leq 2m-2, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

These are the coefficients of $(1 + w + \cdots + w^{m-1})^2$. The numerator in Proposition 1.10, equations (1.11)–(1.12), of the published source [1] is the ordinary two-variable polynomial

$$P_{m,n}(s,t) = \sum_{a=0}^{2m-2} \sum_{b=0}^{2n} D_m(a) D_m(mb + na + m + n - 1 - 2mn) s^a t^b. \tag{2}$$

The source defines

$$Q_{m,n}(z) = z^{1-2n} P_{m,n}(z,z). \tag{3}$$

It proves that this is a polynomial of degree $2k$ with palindromic coefficients. The apparent negative power in (3) cancels. The formula (2) is Almughrabi's earlier Proposition 2 [2]; Edholm–Mathew also give an equivalent five-piece formula with explicit ceiling functions in Theorem 1.3.

Theorem 1. *For every coprime pair $m > n \geq 1$ and every complex number z with $|z| = 1$, one has $Q_{m,n}(z) \neq 0$. Exactly $m-n$ roots of $Q_{m,n}$ lie in $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, counting multiplicities.*

This is the full assertion of the published Conjecture 4.20 [1], recorded as MathDB 368572. The source already establishes differences $k = 1, 2, 3, 4, 6$, and describes infinitely many explicit pairs for every fixed difference. Those known cases are included in the uniform theorem and receive no separate discovery claim.

mathdb:2610.00770v1 28 Oct 2026

2 A finite sum of squares

For each integer $0 \le r < m$, define

$$A_r(z) = \sum_{p=0}^{m-1} z^{e_r(p)}, \qquad e_r(p) = p - \left\lfloor \frac{np+r}{m} \right\rfloor. \tag{4}$$

The exponents satisfy $0 \le e_r(p) \le k$. Indeed, $np + r < m(p+1)$ gives the lower bound, and $e_r(p) = \lceil (kp-r)/m \rceil \le k$ gives the upper bound.

Lemma 2. *For every nonzero complex z,*

$$Q_{m,n}(z) = z^k \sum_{r=0}^{m-1} A_r(z) A_r(z^{-1}). \tag{5}$$

Consequently, if $|z| = 1$, then

$$z^{-k} Q_{m,n}(z) = \sum_{r=0}^{m-1} |A_r(z)|^2. \tag{6}$$

Proof. We give a direct coefficient proof, keeping the source normalization. For $a, c \in \{0, \dots, 2m-2\}$, the substitution

$$c = mb + na + m + n - 1 - 2mn$$

identifies the nonzero terms of (2) with the congruence $c - na \equiv n - 1 \pmod m$. Conversely that congruence determines the integer b, and $0 \le b \le 2n$: the numerator determining mb is strictly between $-m$ and $m(2n+1)$, and is divisible by m. Thus (3) becomes

$$Q_{m,n}(z) = \sum_{\substack{0 \le a,c \le 2m-2 \\ c - na \equiv n-1 \pmod m}} D_m(a) D_m(c)\, z^{(ka+c-n+1)/m}. \tag{7}$$

The exponent is an integer; the same substitution shows that it equals $a + b + 1 - 2n$.

The integer $D_m(a)$ counts the ordered pairs $p, p_2 \in \{0, \dots, m-1\}$ with $p + p_2 = a$. Complementing the second member writes $a = p + m - 1 - p'$ with $0 \le p, p' < m$. Similarly write $c = q + m - 1 - q'$ with $0 \le q, q' < m$. The congruence in (7) becomes

$$q - np \equiv q' - np' \pmod m. \tag{8}$$

Group these pairs by their common residue $r \in \{0, \dots, m-1\}$. For a fixed p and r, there is exactly one $q \in \{0, \dots, m-1\}$, namely

$$q = np + r - m \left\lfloor \frac{np+r}{m} \right\rfloor.$$

The analogous formula holds for q'. The exponent in (7) now equals

$$\frac{k(p + m - 1 - p') + q + m - 1 - q' - n + 1}{m} = k + e_r(p) - e_r(p').$$

All changes of variables above are bijections of finite index sets, including their multiplicities $D_m(a)D_m(c)$. Summing first over p, p' and then over r proves (5). For unit z, the real coefficients of A_r give $A_r(z^{-1}) = \overline{A_r(z)}$, proving (6). □

3 Strict positivity and root allocation

Lemma 3. *The two polynomials A_0 and A_1 have no common nonzero complex root.*

Proof. Coprimality gives a unique $p_* \in \{1, \dots, m-1\}$ with $np_* \equiv m-1 \pmod m$. Write $np_* = mh + m - 1$. In passing from $r = 0$ to $r = 1$ in (4), the floor increases only for this one value of p. Consequently

$$A_1(z) - A_0(z) = z^{p_* - h - 1}(1 - z). \tag{9}$$

The exponent is nonnegative: the general exponent bound for A_1 gives $h + 1 \le p_*$. If $z \neq 0, 1$, the right side of (9) is nonzero. At $z = 1$, one has $A_0(1) = m \neq 0$. □

Proof of Theorem 1. On the unit circle every summand in (6) is nonnegative, and Lemma 3 makes at least one of the first two summands strictly positive. Hence $z^{-k}Q_{m,n}(z) > 0$ and $Q_{m,n}(z) \neq 0$.

For completeness, the same coefficient identity recovers the algebra needed to count roots. Because $0 \le e_r(p) \le k$, every term on the right of (5) has exponent in $[0, 2k]$. The expression is an ordinary polynomial with nonnegative integer coefficients, and swapping p and p' shows that it is palindromic. In A_0, the indices $p = 0$ and $p = m - 1$ give exponents 0 and k, respectively. Thus both the constant and degree-$2k$ coefficients of $Q_{m,n}$ are positive. The degree is exactly $2k$ and zero is not a root. Reciprocity pairs each root ζ with ζ^{-1} with the same multiplicity. Since there is no unit-circle root, inversion gives a multiplicity-preserving bijection between roots inside and outside the unit disc. The fundamental theorem of algebra counts $2k$ roots in all, so exactly k lie in $\mathbb{D}^*$. □

4 Relation to the older kernel formulas

The finite roots-of-unity machinery underlying this calculation is classical. In particular, the weighted geometric-square identity itself already appears in Almughrabi [2, eq. (24)]; it is not a new claim of this paper. To make the relation precise, put $S_m(u) = 1 + u + \cdots + u^{m-1}$ and let ω be a primitive mth root of unity. For $b \neq 0$, root-of-unity filtering gives

$$\begin{aligned} &b^{m+n-1-2mn} P_{m,n}(ab^n, b^m) \\ &\quad = \frac{1}{m} \sum_{j=0}^{m-1} \omega^{(1-n)j} S_m(a\omega^{-nj})^2 S_m(b\omega^j)^2. \end{aligned} \tag{10}$$

The identity is Almughrabi's formula after changing j to $-j$ and matching its normalization. For unit u, $S_m(u)^2 = u^{m-1}|S_m(u)|^2$. The phases in each summand of (10) therefore cancel, yielding a nonnegative sum. The direct finite reindexing in Lemma 2 gives the explicit floor polynomials and the strictness certificate (9), including all resonant roots of unity.

Chakrabarti–Konkel–Mainkar–Miller [3] supply the earlier triangular-coefficient kernel formula, and Almughrabi [2] simplifies it in two complex dimensions. These formulas receive full prior credit. Neither the actual earlier formula papers nor the June 2026 source examined in this review asserts the full boundary positivity conclusion proved here. The source explicitly retains Conjecture 4.20 and mentions a further paper in preparation; no public full version of that successor was located. This bounded comparison supports the proposed source-conjecture resolution and does not certify absolute worldwide priority.

5 The complex-analytic consequence and formal scope

Edholm–Mathew's source formula is

$$K_{m/n}(Z, W) = \frac{P_{m,n}(s, t)}{m\pi^2(1 - t)^2(t^n - s^m)^2}, \qquad s = Z_1\overline{W_1}, \quad t = Z_2\overline{W_2},$$

mathdb:2610.00770v1 28 Oct 2026

on $\mathbb{H}_{m/n} = \{(Z_1, Z_2) : |Z_1|^{m/n} < |Z_2| < 1\}$. The theorem supplies roots $z \in \mathbb{D}^*$ of $P_{m,n}(z, z)$. The diagonal (z, z) belongs to $\mathbb{H}_{m/n}$, and the denominator there is nonzero. Such a pair of products is realized by taking $Z = (\sqrt{|z|}e^{i \arg z}, \sqrt{|z|}e^{i \arg z})$ and $W = (\sqrt{|z|}, \sqrt{|z|})$, both in the domain. Thus the source kernel formula gives zeros for every rational parameter $m/n > 1$. This is the source's stated consequence of its polynomial conjecture, not a separate counted result. The passage uses the published kernel formula; the Lean certificate concerns the explicit polynomials and finite algebra, not the theory constructing Bergman kernels or proper holomorphic maps.

The ten-module Lean certificate verifies the signed source coefficient D_m, its support and literal double-sum normalization, the affine coefficient reindexing, triangular convolution, complementary-pair and residue reindexings, the exact unit-circle sum-of-squares identity, the modular carry and strictness argument, and nonvanishing for every coprime pair. It also constructs an ordinary polynomial over the natural numbers, maps it into $\mathbb{C}[X]$, proves equality with the literal source expression at every nonzero argument, bounds every selected integer exponent before conversion to a natural exponent, and proves exact degree $2k$, nonzero constant coefficient and reciprocity. Full complex factorization proves the reciprocal-root multiset identity. The final source endpoint therefore certifies both unit-circle nonvanishing and exactly k punctured-disc roots, including all multiplicities. No source, SOS, degree, reciprocity or root-count premise is left unproved.

The main endpoint is `source_full_conjecture_368572` in namespace `BergmanUnitCircle`. Fresh pinned replay checks all 78 local declarations with warnings treated as errors. Axiom inspection permits only `propext`, `Classical.choice`, and `Quot.sound`. The compiler is Lean 4.34.1; exact compiler and mathlib commits, respectively, are

`5045d0056413266e57c625dcd7c365b10e377c52`
`d13f23b723b8a846827a245b89c10fc7d3f11612.`

The packet contains exact dependency pins, independent reviews, all-declaration replay receipts and types/axioms. No numerical experiment, proof hole, added axiom, unsafe declaration or native computation assumption is used. The general Laurent identity above is proved mathematically; the formally certified SOS specializes it to unit inputs, which suffices for the full stated source conclusion.

For a separate arithmetic cross-check, the literal ceiling-function five-piece formula in the source was constructed with exact integer polynomials for every coprime pair $2 \le m \le 80$. All $1,965$ cases agree with (5) and (9). These checks corroborate the definitions and are not assumptions in the uniform proof.

References

[1] L. D. Edholm and V. T. Mathew, *Arithmetic properties and zeros of the Bergman kernel on a class of quotient domains*, Mathematische Zeitschrift **313** (2026), article 43, doi:10.1007/s00209-026-04059-8; also arXiv:2505.20489v3.

[2] R. Almughrabi, *Bergman kernels of two dimensional monomial polyhedra*, Complex Analysis and Operator Theory **17** (2023), article 103, doi:10.1007/s11785-023-01401-4; also arXiv:2303.14268.

[3] D. Chakrabarti, A. Konkel, M. Mainkar and E. Miller, *Bergman kernels of elementary Reinhardt domains*, Pacific Journal of Mathematics **306** (2020), 67–93; arXiv:1909.03164.

mathdb:2610.00770v1 28 Oct 2026