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Four interior roots in Yuster's entropy polynomial

Research proof and formalization packet

October 26, 2026

Abstract

We disprove the polynomial-root conjecture in Yuster's *Almost k-union closed set systems*: at $k = 48$, its canonically parameterized polynomial has at least four distinct roots in $(0, 1)$. Five exact rational sign checks, together with rational bounds on the canonical algebraic parameter, provide a short certificate. Lean verifies the literal source coefficients, the parameter, every sign, the four real roots, and the resulting root count with multiplicity. No claim is made about the smallest counterexample or the exact total number of roots. The underlying entropy inequality is a different statement and has already been proved in subsequent work.

1 The exact source polynomial

Fix an integer $k \geq 2$. Following Yuster [1, Section 3], let ψ_k be the unique solution in $[0, 1]$ of

$$(1 - \psi_k)^k = \psi_k, \qquad \phi_k = 1 - \psi_k, \qquad \alpha_k = \phi_k^{k-1}.$$

The coefficients in Lemma 3.5 of that source are determined by

$$\begin{aligned} C(k, 0, 0) &= \frac{1}{(k-1)!}, \\ C(k, 0, j) &= 0 \quad (j > 0), \qquad C(k, t, 0) = 0 \quad (t > 0), \\ C(k, t, j) &= (kj - t + 1)C(k, t-1, j) + kC(k, t-1, j-1) \quad (t, j > 0). \end{aligned}$$

For concise display, set $G_k(x) = \sum_{i=0}^{k-1} x^i$. Corollary 3.7 defines $p_k(x) = \alpha_k \rho_k(x) - \sigma_k(x)$, where

$$\begin{aligned} \rho_k(x) = \sum_{j=0}^{k-1} &(-1)^j j! C(k, k+1, j+2) \\ &\cdot \left[(x^k - 1)^{k-j-1} x^{kj+k} - (x^k - 1)^k \right], \end{aligned} \tag{1}$$

$$\begin{aligned} \sigma_k(x) = \sum_{j=0}^{k-1} &(-1)^j \binom{k+1}{j+2} \\ &\cdot \left[x^{j+1} (x-1)^{k-j-1} G_k(x)^k - (x^k - 1)^k \right]. \end{aligned} \tag{2}$$

Conjecture 3.9 asserts that p_k has at most two real roots in $(0, 1)$, counted with multiplicity, for every $k \geq 2$. This is the assertion recorded as MathDB 357503.

Theorem 1. *The polynomial p_{48} defined above has at least four distinct roots in $(0, 1)$. In particular its number of interior roots, counted with multiplicity, is at least four, and Yuster's Conjecture 3.9 is false.*

2 A rational sign certificate

Let

$$a = \frac{61}{1000}, \qquad b = \frac{613}{10000}.$$

First, the source parameter satisfies $a < \alpha_{48} < b$. Indeed $\phi_k + \phi_k^k = 1$ and $\phi_k > 0$, so $\phi_k(1+\alpha_k) = 1$ and hence

$$\alpha_k(1+\alpha_k)^{k-1} = 1. \tag{3}$$

Exact rational arithmetic gives

$$a(1+a)^{47} < 1 < b(1+b)^{47}.$$

The function $u \mapsto u(1+u)^{47}$ is strictly increasing on $[0,\infty)$, proving the asserted bounds from (3). Conversely the certificate constructs the unique source ψ_{48} in $[0,1]$; it does not assume an arbitrary coefficient in this bracket.

The following table records exact rational signs, using the literal finite sums (1)–(2). A minus or plus denotes a strict inequality, not a floating-point approximation.

x	$\rho_{48}(x)$	endpoint expression	its sign
$1/100$	< 0	$a\rho_{48}(x) - \sigma_{48}(x)$	$-$
$1/10$	< 0	$b\rho_{48}(x) - \sigma_{48}(x)$	$+$
$1/2$	< 0	$a\rho_{48}(x) - \sigma_{48}(x)$	$-$
$3/5$	< 0	$b\rho_{48}(x) - \sigma_{48}(x)$	$+$
$4/5$	< 0	$a\rho_{48}(x) - \sigma_{48}(x)$	$-$

For reproducibility, one can calculate all entries using integer rows $B_{t,j} = 47!C(48,t,j)$, starting from $B_{0,0} = 1$ and applying

$$B_{t+1,j} = (48j - t)B_{t,j} + 48B_{t,j-1}.$$

Undefined row entries are zero. Divide by 47! when substituting in (1); all ensuing evaluations are rational. The packet contains the full rational values, and Lean independently checks the inequalities by kernel reduction of exact rational arithmetic.

Since $\rho_{48}(x) < 0$ at every listed point, increasing the parameter decreases $\alpha\rho_{48}(x) - \sigma_{48}(x)$. Thus the bounds $a < \alpha_{48} < b$ transfer the five signs in the table to p_{48}:

$$p_{48}(1/100) < 0,\ p_{48}(1/10) > 0,\ p_{48}(1/2) < 0,\ p_{48}(3/5) > 0,\ p_{48}(4/5) < 0.$$

Polynomial continuity and the intermediate value theorem give a root in each of the four open intervals

$$(1/100, 1/10), \quad (1/10, 1/2), \quad (1/2, 3/5), \quad (3/5, 4/5).$$

The intervals are pairwise disjoint and contained in $(0,1)$, so these four roots are distinct. The polynomial is nonzero by its first strict sign. Each of the four roots therefore occurs with positive multiplicity in its root multiset. This proves Theorem 1.

3 Formal coverage and reproducibility

The seven modules `YusterCoefficients`, `YusterPolynomial`, `YusterArithmetic`, `YusterAlpha`, `YusterSigns`, `FourRoots`, and `YusterRootCounterexample` contain 53 local declarations. The source recurrence, including its factorial normalization, is proved for the implemented exact coefficients. The real polynomial is literally (1)–(2); no auxiliary proxy polynomial or unproved derivative

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identity is assumed. The source parameter is constructed and shown unique in its specified interval. Rational comparisons are checked with ordinary kernel evaluation (`decide +kernel`), not a native computation axiom. Real sign transfer, intermediate roots, distinctness, and the multiset count are also certified.

In namespace `YusterCounterexample`, the main theorem `source_rootcount_ge_four` counts the roots of the actual polynomial with its canonical parameter. The further endpoint `source_conjecture_357503_false` negates the source's universal statement with its defining equation for ψ_k. No sign, canonical-parameter, root-existence, or source-identity assumption is left to the user.

A fresh isolated replay compiled all declarations with warnings as errors under Lean 4.34.1 and the pinned mathlib dependencies. The compiler and mathlib commits, respectively, are

`5045d0056413266e57c625dcd7c365b10e377c52`
`d13f23b723b8a846827a245b89c10fc7d3f11612.`

Axiom inspection found only `propext`, `Classical.choice`, and `Quot.sound`. The packet includes the replay script, exact dependency pins, declaration types and axiom inventory, and independent review. No proof holes or added axioms are used.

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4 Relation to subsequent work and novelty limits

This counterexample concerns the proposed polynomial root bound. It does not refute an entropy inequality or a union-closed-set theorem. Ho [3] proves the generalized entropy inequality by another route; that result is credited and is not claimed here. Wakhare [2, Conjecture 2] proposes a closely related polynomial root bound. The packet claims one outcome, and does not present a separate Lean refutation of Wakhare's formulation.

The source review on October 26, 2026 inspected Yuster's complete primary manuscript, including its actual coefficient formulas and Conjecture 3.9. Only revision v1 was listed. MathDB's references field contains an inconsistent "solved" description quoting an already resolved $k = 2$ entropy constant from the source introduction; it is not a resolution of the root conjecture on page 9. No posted solution of the root assertion was present in the reviewed record.

Targeted primary-source searches and independent review located no earlier root-count counterexample. The complete later Yuster–Yashfe manuscript was not located, and the published Wakhare body was not separately compared with its latest arXiv v2. These are explicit limits on the prior-art assessment. The result is offered with a documented, bounded novelty check, not an exhaustive worldwide priority claim. Neither the exact total number of roots nor the minimality of $k = 48$ is asserted.

References

[1] R. Yuster, *Almost k-union closed set systems*, arXiv:2302.12276v1 (2023), Lemma 3.5, Corollary 3.7, Conjecture 3.9.

[2] T. Wakhare, *Iterated Entropy Derivatives and Binary Entropy Inequalities*, Journal of Approximation Theory **307** (2025), 106143; primary manuscript arXiv:2312.14743v2, Conjecture 2.

[3] B.S. Ho, *A generalization of Boppana's entropy inequality*, arXiv:2601.19327v1 (2026).