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An odd-degree identity for polynomial values at derivative roots

Research proof and formalization packet

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Abstract

We prove Conjecture 2 of Wojnar, Wojnar and Brin: an alternating binomial combination of the means of an odd-degree polynomial evaluated at the roots of its successive derivatives vanishes. The result holds over the complex numbers, with roots counted with multiplicity. Normalized Newton moments extend these means polynomially in the remaining derivative degree. Their artificial zero-degree value is an apolar self-pairing, which vanishes in odd degree, so a finite difference gives the identity. The full arbitrary-polynomial statement, including repeated roots and the source definitions, is verified in Lean. Classical moment, finite-difference and apolarity ingredients are explicitly credited.

1 The exact identity

Let $f \in \mathbb{C}[z]$ have degree $D \geq 1$. For $0 \leq r < D$ write

$$M_r(f) = \frac{1}{D-r} \sum_{f^{(r)}(\zeta)=0} f(\zeta),$$

where the sum uses the full multiset of $D-r$ roots, including multiplicity. This is the source mean denoted $\varphi_{D,0,r}$ by Wojnar–Wojnar–Brin [1].

Theorem 1. *For every complex polynomial of odd degree D,*

$$\sum_{r=1}^{D-1} (-1)^r \binom{D}{r} M_r(f) = 0. \tag{1}$$

This is Conjecture 2 of the latest listed source revision, arXiv:1706.08381v2, Section 4, and MathDB 336466. No real-rootedness, genericity or simple-root assumption is needed. For $D = 1$ the sum is empty. Known individual degrees and the auxiliary identities below are not separately counted outcomes.

2 Polynomial continuation of normalized moments

First assume f is monic. Write

$$f(z) = F_D(z) = \sum_{k=0}^{D} \binom{D}{k} a_k z^{D-k}, \qquad a_0 = 1,$$

and extend the parameter sequence beyond D arbitrarily. For each integer $m \geq 1$ put

$$F_m(z) = \sum_{k=0}^{m} \binom{m}{k} a_k z^{m-k}.$$

For $1 \leq m \leq D$, direct differentiation gives

$$f^{(D-m)} = \frac{D!}{m!} F_m. \tag{2}$$

Thus F_m has exactly the required derivative-root multiset. Write $S_i(m)$ for the sum of the ith powers of its roots.

Define polynomials in an independent variable T by

$$B_i(T) = \frac{(T-1)(T-2)\cdots(T-i+1)}{i!}, \qquad C_i(T) = TB_i(T) \quad (i \geq 1),$$

with empty product one. Set $q_0(T) = 1$ and, for $i \geq 1$,

$$q_i(T) = -\sum_{k=1}^{i-1} C_k(T) a_k q_{i-k}(T) - iB_i(T) a_i. \tag{3}$$

Lemma 2. *For every integer $m \geq 1$ and $i \geq 1$,*

$$q_i(m) = \frac{S_i(m)}{m}, \qquad \deg q_i \leq i-1, \qquad q_i(0) = (-1)^i a_i.$$

Proof. The coefficients of F_m are $c_k = \binom{m}{k} a_k$, with $c_k = 0$ for $k > m$. Newton's identities, including the homogeneous recurrence beyond the degree, give for every $i \geq 1$

$$S_i(m) + \sum_{k=1}^{i-1} c_k S_{i-k}(m) + ic_i = 0.$$

Since $C_i(m) = \binom{m}{i}$ and $mB_i(m) = C_i(m)$, division by m and induction prove the evaluation claim. This also covers $i > m$, which is essential when evaluating a degree-D polynomial on roots of F_m. In (3), each product $C_k q_{i-k}$ has degree at most $k + (i-k-1) = i-1$, as does B_i. Induction proves the degree bound. Finally, $C_k(0) = 0$ and $B_i(0) = (-1)^{i-1}/i$, yielding the stated value at zero. □

Now form

$$\Phi(T) = \sum_{k=0}^{D} \binom{D}{k} a_k q_{D-k}(T).$$

Lemma 2, with $q_0 = 1$, gives $\deg \Phi \leq D-1$ and

$$\Phi(m) = \frac{1}{m} \sum_{F_m(\zeta)=0} f(\zeta) = M_{D-m}(f) \qquad (1 \leq m \leq D). \tag{4}$$

In particular $\Phi(D) = 0$, because f vanishes at every root of $F_D = f$. At the artificial argument zero,

$$\Phi(0) = \sum_{k=0}^{D} \binom{D}{k} (-1)^{D-k} a_k a_{D-k}. \tag{5}$$

This is a value of the polynomial continuation, not a mean over roots of a constant polynomial. For odd D, pairing the k and $D-k$ terms in (5) shows that $\Phi(0) = 0$. This is the classical odd-degree self-apolarity cancellation.

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3 Finite differences and the source statement

Proof of Theorem 1. The Dth finite difference of a polynomial of degree at most $D-1$ vanishes, hence

$$\sum_{m=0}^{D}(-1)^m\binom{D}{m}\Phi(m)=0.$$

Both endpoint terms are zero. Substitute (4) in the remaining sum and reindex $r=D-m$. The resulting sum is $(-1)^D$ times (1); this nonzero common factor can be removed.

For arbitrary f, divide by its nonzero leading coefficient c to obtain a monic polynomial g. Every derivative-root multiset is unchanged by this scaling, and $M_r(f)=cM_r(g)$. Multiplying the monic identity by c proves the full statement. □

4 Classical ingredients, novelty, and formal scope

The normalized root-moment and finite-difference phenomenon has important older precedents. Toda [2] studies finite differences of test-function means over successive derivative roots; his pages 315–316 use binomial-normalized coefficients and Waring's formula. Popoviciu [3] explicitly records Toda's vanishing result for a test polynomial of degree at most n, with n strictly smaller than the degree of the root polynomial. Those are close precedents for the moment-continuation ingredient and receive credit here.

The present target has the same-degree self-test and an omitted zero-degree root mean. The needed boundary value is supplied by the apolar continuation (5); the inspected older statements do not explicitly give this endpoint. Odd self-apolarity itself is also classical, for example Ottaviani [4], Section 4.3, Proposition 18. The proposed contribution is the complete source-mean bridge and resulting all-degree resolution of [1], not any of these classical tools or an exhaustive classification of all mean relations.

The detailed primary-source investigation found no previous full proof of the exact source conjecture. Toda's 1934 precursor [5] remained inaccessible in full because its public publisher endpoint returned an anti-bot page. Its 1936 successor and Popoviciu's original 1940 discussion were read at the formula level, including rendered pages, but do not exclude every possible additional formula in the unread precursor. This material retrieval limitation is explicit in the independent novelty review. Acceptance under the documented no-known-prior-resolution standard is not a certification of worldwide priority.

The six-module Lean certificate inventories 50 declarations. It proves Newton's identities on complex root multisets, all moment continuation claims including indices above the remaining degree, exact iterated derivative/root correspondence, endpoint cancellation, finite differences, interval trimming and arbitrary-polynomial normalization. The final theorem `OddRootMeans.source_conjecture_336466` assumes only that the complex polynomial's natural degree is odd. No user-supplied moment, splitting, source-bridge or distinct-root hypothesis remains. Fresh independent replays use pinned Lean 4.34.1 and only `propext`, `Classical.choice`, and `Quot.sound`; no proof holes or additional axioms occur. Numerical trace experiments are supplementary and not dependencies of the proof.

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References

[1] G. G. Wojnar, D. Sz. Wojnar and L. Q. Brin, *Universal Linear Mean Relationships in All Polynomials*, arXiv:1706.08381v2 (2017), Section 4, Conjecture 2; root conventions in Section 1.

[2] K. Toda, *A Method of Approximation of Convex Functions*, Tohoku Math. J. 42 (1936), 311–317. Primary text.

[3] T. Popoviciu, *Notes sur les fonctions convexes d'ordre supérieur (III)*, Mathematica 16 (1940), 74–86, Section 11. Institute-hosted original.

[4] G. Ottaviani, *Five Lectures on Projective Invariants*, arXiv:1305.2749 (2013), Section 4.3, Proposition 18.

[5] K. Toda, *On Certain Functional Inequalities*, J. Sci. Hiroshima Univ., Ser. A 4 (1934), 27–40. DOI: 10.32917/hmj/1558749761. Full body not inspected; see the stated novelty limitation.

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