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# A sharp converse fractional-binomial inequality through log-convexity

Research proof and formalization packet

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**Abstract**

We give a proof of the converse fractional-binomial inequality posed by Gerhold and Simon as Conjecture 3.2. A finite sum of scaled Gamma quotients is log-convex in the scaling parameter. Its exact values at parameters one and two then give the desired sharp lower bound, and a stronger bound depending on the two input weights. The Gamma-quotient convexity is classical; we also give a finite-grid proof suited to formal verification. A companion Lean certificate verifies the source inequality, the biased refinement, and optimality. The prior-art review records the known ingredients and the limitations of the novelty investigation.

## 1 Statement and relation to the source question

For $k \in \mathbb{N}_0$, $a > 0$, and $x, y \geq 0$, put

$$S_k(a; x, y) = \sum_{j=0}^{k} \frac{\Gamma(ak+1)}{\Gamma(aj+1)\Gamma(a(k-j)+1)} x^{aj} y^{a(k-j)}.$$

We use $0^0 = 1$. The quotient agrees with the generalized binomial coefficient in the primary source.

**Theorem 1.** *For every $k \in \mathbb{N}_0$, $a \geq 2$, and $x, y \geq 0$,*

$$(x+y)^{ak} \leq 2^{a-1} S_k(a; x, y). \tag{1}$$

*If $x + y > 0$, set $p = x/(x+y)$ and $q = y/(x+y)$. Then the stronger estimate*

$$S_k(a; x, y) \geq (x+y)^{ak} \left( \frac{1 + (q-p)^{2k}}{2} \right)^{a-1} \tag{2}$$

*holds. The coefficient $2^{a-1}$ in (1) is optimal for each fixed $a \geq 2$, uniformly over $k, x, y$.*

For $a > 2$ and positive integer $k$, (1) is exactly Gerhold–Simon Conjecture 3.2 [1], recorded as MathDB 352508. The endpoint $a = 2$, the zero cases, and the sharpness example are included to specify the full statement; they are not separate proposed discoveries. The biased bound is a consequence of the same argument and is likewise part of this single proposed result.

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## 2 A finite-grid proof of the classical convexity ingredient

The log-convexity of the individual scaled Gamma quotients is classical. Alzer's complete-monotonicity theorem concerns the corresponding scaled binomial probabilities [2]; Qi and Guo explicitly record their logarithmic complete monotonicity in Section 5.2, equation (5.1) [3]. The following elementary proof is included to make all analytic dependencies available to a Lean certificate.

**Lemma 2.** *If $j, l$ are positive integers and $\phi : [0,1] \to \mathbb{R}$ is decreasing, then*

$$\sum_{r=1}^{j} \phi(r/j) + \sum_{s=1}^{l} \phi(s/l) \le \sum_{t=1}^{j+l} \phi(t/(j+l)).$$

*Here "decreasing" permits equality.*

*Proof.* Define integer ranks

$$L(r) = r + \lfloor lr/j \rfloor, \qquad R(s) = s + \lfloor (js-1)/l \rfloor.$$

Each map is strictly increasing and takes its indicated domain into $\{1, \dots, j+l\}$. Their images are disjoint. Indeed, if $L(r) = R(s)$, write $u = \lfloor lr/j \rfloor$ and $v = \lfloor (js-1)/l \rfloor$, so $r + u = s + v$. If $u \ge s$, then $v \ge r$ and $js \le ju \le lr \le lv < js$, a contradiction. If $u < s$, then $v < r$ and $lr < j(u+1) \le js \le l(v+1) \le lr$, again a contradiction. The images therefore partition $\{1, \dots, j+l\}$ by cardinality. Moreover,

$$jL(r) \le (j+l)r, \qquad lR(s) < (j+l)s.$$

Consequently $L(r)/(j+l) \le r/j$ and $R(s)/(j+l) < s/l$. Apply the decreasing property and sum over the partition. □

**Lemma 3.** *For nonnegative integers $j, l$, the function*

$$a \longmapsto \log\Gamma((j+l)a+1) - \log\Gamma(ja+1) - \log\Gamma(la+1)$$

*is convex on $(0, \infty)$.*

*Proof.* If either integer is zero, the function is identically zero. Suppose both are positive. For a positive integer $d$ and $N \ge 1$ define

$$A_{d,N}(a) = da \log(Nd) + \log((Nd)!) - Nd \log d \\ - \sum_{i=0}^{N-1} \sum_{r=1}^{d} \log(a + i + r/d).$$

Euler's Gamma limit gives, for every $a > 0$,

$$A_{d,N}(a) \longrightarrow \log\Gamma(da+1).$$

This follows directly by taking logarithms in $\Gamma(z+1) = \lim_{M\to\infty} M! M^z / \prod_{m=1}^{M}(z+m)$, with $z = da$ and $M = Nd$, and grouping the denominator into blocks. Twice differentiating the finite expression gives

$$A''_{d,N}(a) = \sum_{i=0}^{N-1} \sum_{r=1}^{d} \frac{1}{(a + i + r/d)^2}.$$

For each $i$, apply Lemma 2 to $\phi(t) = (a + i + t)^{-2}$. It follows that $A''_{j+l,N} - A''_{j,N} - A''_{l,N} \ge 0$. Thus $A_{j+l,N} - A_{j,N} - A_{l,N}$ is convex on $(0, \infty)$. Passing to the pointwise limit in the defining convexity inequality proves the assertion. No differentiation of an infinite series or integral is needed. □

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## 3 Proof of the sharp lower bound

First suppose $x, y > 0$. Normalize $p = x/(x+y)$, $q = y/(x+y)$, so $p, q > 0$ and $p + q = 1$, and define

$$F(a) = \sum_{j=0}^{k} \frac{\Gamma(ak+1)}{\Gamma(aj+1)\Gamma(a(k-j)+1)} p^{aj} q^{a(k-j)}, \qquad a > 0.$$

Each summand has convex logarithm: apply Lemma 3 with $l = k - j$ and add the affine function $aj \log p + a(k-j) \log q$. A finite sum of positive log-convex functions is log-convex. Explicitly, for $0 < t < 1$, the termwise convexity inequalities followed by Hölder give

$$F((1-t)a + tb) \le \sum_j f_j(a)^{1-t} f_j(b)^t \le F(a)^{1-t} F(b)^t.$$

The cases $t = 0, 1$ are equalities.

The ordinary binomial theorem and its even-part identity give exact anchors

$$F(1) = (p+q)^k = 1, \qquad F(2) = \frac{(p+q)^{2k} + (q-p)^{2k}}{2} = \frac{1 + (q-p)^{2k}}{2} \ge \frac{1}{2}.$$

For $a \ge 2$, let $t = 1/(a-1) \in (0, 1]$. Since $2 = (1-t) \cdot 1 + t \cdot a$, convexity of $\log F$ yields

$$\log F(2) \le (1-t) \log F(1) + t \log F(a) = t \log F(a).$$

Therefore

$$F(a) \ge F(2)^{a-1} = \left( \frac{1 + (q-p)^{2k}}{2} \right)^{a-1} \ge 2^{1-a}.$$

Multiplying by $(x+y)^{ak}$ proves both estimates for positive $x, y$.

For $k = 0$, $S_0 = 1$ and both conclusions follow immediately. For $k > 0$, if one input is zero, the sole nonzero summand is the endpoint corresponding to the other input, so $S_k = (x+y)^{ak}$; the biased factor is then one when $x + y > 0$. If both inputs vanish, both sides of (1) vanish. This covers all boundary cases. Finally, for $k = 1$ and $x = y > 0$, $S_1 = 2x^a$ and $(x+y)^a = 2^a x^a$. Equality holds in (1), proving optimality. □

## 4 Prior bounds, novelty limits, and formal verification

The proposed contribution is the full sharp finite-sum lower bound and its biased version, not the classical Gamma-ratio convexity. The original source already identifies the sharp coefficient using $k = 1, x = y$. Hino–Namba's published 2026 Theorem 2.9 [4] gives, for $a \ge 2$,

$$a^{1-a^2} \le Z_{a,p}^{(k)} \le a, \qquad Z_{a,p}^{(k)} = aF(a).$$

The present argument gives $Z_{a,p}^{(k)} \ge a 2^{1-a}$, a strictly stronger uniform lower bound. No claim that a short deduction is novel merely because it has been formalized is intended. The separate review record documents the primary sources searched and any remaining overlap risks.

The formal construction uses the same Euler-limit route as Lemma 3. The rank partition, real grid comparison, finite derivatives, Gamma limit, finite log-sum convexity, exact anchors, and homogeneity are separate proof modules. The final source endpoint quantifies over all $k \in \mathbb{N}_0$, $a \ge 2$, and $x, y \ge 0$, with no additional convexity or grid hypothesis. The packet's verification

receipt records all 67 declarations in seven modules, source hashes, compiler and dependency pins, and every declaration's axiom report. The certificate uses Lean 4.34.1 with pinned Mathlib and no proof holes or additional axioms; the only reported axioms are `propext`, `Classical.choice`, and `Quot.sound`. Independent mathematical, source, and formal reviews accompany the packet. No prior full sharp bound was found in the inspected literature; this is a documented novelty screen, not an absolute certification of worldwide priority. Alzer's complete article could not be retrieved; its theorem and corollary types were checked through accessible primary successors, as detailed in the review.

## References


[1] S. Gerhold and T. Simon, *A converse to the neo-classical inequality with an application to the Mittag-Leffler function*, Monatshefte für Mathematik 200(3) (2023), 627–645, Conjecture 3.2. DOI: 10.1007/s00605-022-01817-8.

[2] H. Alzer, *Complete monotonicity of a function related to the binomial probability*, J. Math. Anal. Appl. 459 (2018), 10–15. DOI: 10.1016/j.jmaa.2017.10.077.

[3] F. Qi and B.-N. Guo, *From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions*, J. Math. Anal. Appl. 493(1), 124478 (online 2020), Section 5.2. Open primary text.

[4] M. Hino and R. Namba, *Fractional binomial distributions induced by the generalized binomial theorem and their applications*, Math. Ann. 395, article 9 (2026), Theorem 2.9. DOI: 10.1007/s00208-026-03424-6.


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