

Projection-body volume for four-dimensional polytopes with at most six facets

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Abstract

For every convex polytope $K \subset \mathbb{R}^4$ with nonempty interior and at most six actual facets, we prove $V_4(\Pi K) \leq (160/3)V_4(K)^3$, with equality precisely for simplices. Here $h_{\Pi K}(u) = V_3(K \mid u^\perp)$ for unit u . An attained simplex-section maximum reduces the six-facet case to a signed sum of four independent exponential variables. A classical log-concavity and Laplace comparison, followed by a two-bracket identity, reduces the estimate to ten polynomial certificates. We give all their coefficients and positive endpoint terms, so repeated knots, zero coefficients and nonsimple polytopes are included. This is a bounded-class result; it gives no theorem for seven or more facets, general four-dimensional bodies, or dimensions five through eight.

Review status. The full proof and exact finite support are presented for independent manuscript review. Independently reviewed Lean support covers only the deterministic algebra of Proposition B; the full geometric theorem is unformalized. This implementation confers no acceptance or release. Original prose: CC BY 4.0; original code: MIT.

1 Statement, context and proof structure

A convex body is compact and convex with nonempty interior. A facet always means an actual irredundant face of codimension one, not a listed inequality in a redundant description. Write V_j for j -dimensional Euclidean volume, and define

$$\mathcal{P}_n(K) = \frac{V_n(\Pi K)}{V_n(K)^{n-1}}, \quad c_n = \frac{(n+1)n^n}{n!}. \quad (1.1)$$

The projection body is unpolarized: its support function on the sphere is $h_{\Pi K}(u) = V_{n-1}(K \mid u^\perp)$, extended homogeneously elsewhere.

Theorem 1. *Every convex polytope $K \subset \mathbb{R}^4$ with nonempty interior and at most six actual irredundant facets satisfies*

$$V_4(\Pi K) \leq \frac{160}{3}V_4(K)^3.$$

Equality holds if and only if K is a simplex. In particular, every exactly-six-facet case is strict, including nonsimple polytopes and all repeated, zero and boundary Gale-coordinate cases.

The simplex upper-bound question originates with Brannen; our historical attribution is through the primary texts actually read, principally Feng–Hu–Liu–Xu (FHLX) [1] and Saroglou [7, 8]. FHLX v1, Theorem 1.3 and Section 4, constructs counterexamples in every $n \geq 9$ using at most $n + 2$

facets, and its introduction leaves dimensions 4 through 8 unresolved. This is a statement about that version's reported context, not a fresh exhaustive open-status survey. Theorem 1 treats only the stated four-dimensional class. It is not a solution of the general Brannen problem.

The pinned OpenAI Source P [2], at revision fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb, supplies the facet, affine and simplex interfaces (`lem:facets`, `lem:affine`, `prop:simplex`). Its product construction is not a proof for all six-facet polytopes. The pinned theorems and their operative proofs are stipulated true in this project; this manuscript does not audit them. The companion Petty manuscript [3] concerns the lower bound and supplies no upper-bound premise here.

The exact section reduction in Sections 2–4 credits FHLX Section 4, especially Propositions 4.1–4.2, and the earlier project round r132, R1–R4 and its normal form $(\star)$ [9]. The latter was a prefrozen exploration with the global estimate unpaid. The contribution assessed in the project is the joint assembly of the conditional-parts Laplace comparison, two-bracket separation and complete four-variable certificate, with geometric strictness. Log-concavity, reliability comparisons, partial fractions and coefficient positivity are not individually claimed as new. In particular the HNBUE and Laplace-transform literature includes Klefsjö [5, 6]; the needed estimate is proved below, without treating those unread bodies as operative premises.

The novelty language is limited: the result is new to this project, and no antecedent was found in the recorded bounded searches. We make no first-result or worldwide-priority claim. The original author and contribution searches were not exhaustive; their exact reading limits are described in Appendix B. No optimal auxiliary constant or all-dimensions positive threshold is claimed.

Here is the chain of the proof. Actual facet data give an attained maximum of sections of a simplex. Complementary minors, linear coarea and the Gamma–Dirichlet change of variables identify it with a density at zero. Integrating two extreme exponential summands leaves four middle coefficients. Conditional log-concavity bounds the resulting Laplace expectation. A deterministic identity reduces its strict upper bound to

$$D(b)\mathbb{P}(W_b > 0)^2 \leq 5\mathbb{E}W_b^+, \quad W_b = \sum_{j=1}^4 b_j E_j, \quad D(b) = \sum_{i < j} |b_i - b_j|, \quad (1.2)$$

for b and $-b$, with strictness for mixed signs. The E_j are independent exponentials of mean one; $w^+ = \max(w, 0)$ and $w^- = \max(-w, 0)$. All ten sign patterns are proved by the explicit appendix certificates.

2 Facet geometry and an attained maximum

We first work in dimension $n \geq 2$. Let K have m actual facets, unit outward normals u_i , positive areas f_i , and supports $h_i = h_K(u_i)$. Translate an interior point to zero, so $h_i > 0$. Put $w_i = f_i u_i$ and let W have these columns. The divergence theorem, applied to constant vector fields and to x/n , gives

$$\sum_i w_i = 0, \quad \sum_i f_i h_i = nV_n(K). \quad (2.1)$$

The normals span $\mathbb{R}^n$: otherwise a nonzero orthogonal vector would be a lineality direction of the intersection of all facet halfspaces, contradicting compactness. Thus $\text{rank } W = n$.

For almost every line parallel to a unit vector u over the projection of K , its two boundary points lie in facet interiors. The exceptional projections of lower-dimensional faces and facets parallel to u have measure zero. Projection of the i th facet has Jacobian $|\langle u_i, u \rangle|$. Counting the two endpoints

proves Cauchy's formula and fixes the normalization:

$$h_{\Pi K}(u) = \frac{1}{2} \sum_i f_i |\langle u_i, u \rangle|, \quad \Pi K = \sum_i [-w_i/2, w_i/2]. \quad (2.2)$$

This is also Source P, `lem:facets`. The segment $[-w_i/2, w_i/2]$ is a translate of $[0, w_i]$; no extra power of two appears in the zonotope volume formula

$$V_n(\Pi K) = \sum_{|J|=n} |\det W_J|. \quad (2.3)$$

To justify it, adding $[0, v]$ extends each nonempty fiber parallel to v by $|v|$, without changing its projection. Fubini gives $V_n(Z + [0, v]) = V_n(Z) + |v|V_{n-1}(Z \mid v^\perp)$, also when Z has lower dimension. Induction on generators and dimension proves (2.3): each projected determinant times $|v|$ is the absolute determinant with v adjoined. Every unordered n -subset occurs once and dependent subsets contribute zero.

We use the following precise form of Minkowski's first inequality:

$$V_n(K)^{(n-1)/n} V_n(L)^{1/n} \leq \frac{1}{n} \sum_i f_i h_L(u_i) \quad (2.4)$$

for compact convex K, L with interiors, with K a polytope. No condition that L contain zero is needed. For clarity, its classical foundation can be reduced to Prékopa's integral inequality [4, Theorem 3]. Applied to indicators of convex sets it gives the multiplicative Brunn–Minkowski inequality. If $a = V(A)^{1/n}$ and $b = V(B)^{1/n}$, apply that inequality to the unit-volume sets $A/a, B/b$ at weights $a/(a+b), b/(a+b)$, then dilate by $a+b$. This proves

$$V(A+B)^{1/n} \geq V(A)^{1/n} + V(B)^{1/n}. \quad (2.5)$$

The first variation at a full-dimensional polytope is

$$\left. \frac{d}{dt} V_n(K + tL) \right|_{0+} = \sum_i f_i h_L(u_i).$$

Here is a local justification. Translate L to contain zero and bound it in a fixed ball. Away from an $O(t)$ neighborhood of the finitely many facet boundaries, an exposed point of L in direction u_i and convexity fill the normal layer of thickness $th_L(u_i)$. Conversely every point of $K + tL$ satisfies the displaced supporting inequalities. The remaining regions lie in $O(t)$ neighborhoods of faces of codimension at least two, with volume $O(t^2)$; the facet boundary strips also have area $O(t)$ and thickness $O(t)$. This proves the derivative. Translating back changes the sum by zero by (2.1). Differentiating (2.5) with $B = tL$ now gives (2.4). These arguments require neither simplicity nor an equality classification for Minkowski's inequality.

Let R be the row space of W and $L_0 = \ker W = R^\perp$. Define

$$H = \{s \in \mathbb{R}^m : \sum_i s_i = 1\}, \quad \Delta = H \cap [0, \infty)^m, \quad S_\lambda = (\lambda + R) \cap \Delta \quad (\lambda \in H).$$

By (2.1), $\mathbf{1} \in L_0$, so $R \subset \mathbf{1}^\perp$.

Lemma 2 (Exact section maximum). *With intrinsic n -dimensional section measure,*

$$V_n(K)^{1-n} = \frac{n^n}{\sqrt{\det WW^T}} \max_{\lambda \in H} \mathcal{H}^n(S_\lambda). \quad (2.6)$$

The maximum is attained at $\lambda_i^* = f_i h_i / (nV_n(K))$, and every $\lambda_i^* > 0$.

Proof. Set $K_\lambda = \{x : \langle w_i, x \rangle \leq \lambda_i \ \forall i\}$. The injective affine map $x \mapsto \lambda - W^T x$ maps this set exactly onto S_λ and has n -Jacobian $\sqrt{\det WW^T}$. Its inverse on the image is continuous, so K_λ is compact when nonempty. Empty or lower-dimensional sections have zero n -volume. Otherwise $h_{K_\lambda}(u_i) \leq \lambda_i/f_i$, and (2.4) gives

$$V_n(K)^{(n-1)/n} V_n(K_\lambda)^{1/n} \leq \frac{1}{n} \sum_i \lambda_i = \frac{1}{n}.$$

This applies even to negative trial supports λ_i . Thus $n^n V_n(K_\lambda) \leq V_n(K)^{1-n}$. Equation (2.1) puts λ^* in H , with positive coordinates, and $K_{\lambda^*} = K/(nV_n(K))$. Its volume realizes the bound, proving both equality and attainment. $\square$

3 Gale coordinates, coarea and exponential normalization

Assume now $m = n + 2$. The kernel is two-dimensional, so $L_0 = \text{span}\{\mathbf{1}, a\}$ for a nonzero $a \perp \mathbf{1}$. Let $I = (\min_i a_i, \max_i a_i)$; it is nonempty. For any vector c , write $D(c) = \sum_{i < j} |c_i - c_j|$.

Lemma 3. $V_n(\Pi K)/\sqrt{\det WW^T} = D(a)/(\sqrt{m}|a|)$.

Proof. Choose an $n \times m$ matrix U with orthonormal rows spanning R . Then $W = MU$ with $M = WU^T$ invertible and $\sqrt{\det WW^T} = |\det M|$. Complete U to an orthogonal matrix Q by the rows $\mathbf{1}/\sqrt{m}$ and $a/|a|$. The complementary-minor identity for an orthogonal matrix gives, for $J^c = \{i, j\}$,

$$|\det U_J| = \left| \det \begin{pmatrix} 1/\sqrt{m} & 1/\sqrt{m} \\ a_i/|a| & a_j/|a| \end{pmatrix} \right| = \frac{|a_i - a_j|}{\sqrt{m}|a|}.$$

One may obtain the identity by taking exterior powers of Q : Hodge duality sends the wedge on J to the complementary wedge, and an orthogonal map commutes with this operation up to its determinant ± 1 . Taking absolute values removes that sign. Equivalently it is Jacobi's cofactor identity using $Q^{-1} = Q^T$. Now (2.3) and $\det W_J = \det M \det U_J$ give the claim. No coordinate differences have been divided out. $\square$

Let s be uniform in Δ with respect to intrinsic volume, $X = \langle a, s \rangle$, and $\sigma(t) = \mathcal{H}^n(\Delta \cap \{\langle a, s \rangle = t\})$. The edge Gram matrix of $\Delta = \text{conv}(e_1, \dots, e_m)$ is $I + \mathbf{1}\mathbf{1}^T$ of determinant m , so $\mathcal{H}^{m-1}(\Delta) = \sqrt{m}/(m-1)!$. The gradient of $\langle a, s \rangle$ in H is a . Resolving Euclidean measure in the direction $a/|a|$ gives the linear coarea density

$$\phi_X(t) = \frac{(m-1)!}{\sqrt{m}|a|} \sigma(t), \quad t \in I. \quad (3.1)$$

It is zero off the closed interval, with arbitrary endpoint values. Every interior t has a barycentric representative with all coordinates positive, and hence a section with positive n -volume. Translate parallel sections into a common n -space. Their convex combinations are contained in the intermediate section; (2.5) makes $\sigma^{1/n}$ concave. It is continuous on I . Section volumes are bounded, since the sections all lie in a fixed ball. Thus we may use a bounded density ϕ_X continuous throughout I , regardless of repeated knots or endpoint section volumes.

For independent $E_i \sim \text{Exp}(1)$ set $S = \sum_i E_i$. In the affine coordinates $s_1, \dots, s_{m-1}$, $E_i = r s_i$ has absolute Jacobian r^{m-1} . The joint density becomes $e^{-r} r^{m-1}$, independent of s on the simplex. Its normalization proves that $s = E/S$ is uniform, independent of S , with

$$S \sim \text{Gamma}(m, 1), \quad \gamma_m(r) = \frac{r^{m-1} e^{-r}}{(m-1)!}, \quad \mathbb{E} S^{-1} = \frac{\Gamma(m-1)}{\Gamma(m)} = \frac{1}{m-1}. \quad (3.2)$$

For $t \in I$ define $Y_t = \sum_i (a_i - t)E_i = S(X - t)$. At least one coefficient is positive and one negative. Their two-term sum has a bounded continuous asymmetric-Laplace density; convolution with the remaining law gives a bounded continuous density f_{Y_t} . Independently the product representation has density

$$\psi_t(y) = \int_0^\infty \phi_X(t + y/r) \frac{\gamma_m(r)}{r} dr.$$

Boundedness and continuity of ϕ_X at t , together with (3.2), give continuity of ψ_t at zero by dominated convergence. The two densities agree almost everywhere and are both continuous there, so

$$f_{Y_t}(0) = \frac{\phi_X(t)}{m-1}, \quad \sigma(t) = \frac{\sqrt{m}|a|}{(m-2)!} f_{Y_t}(0). \quad (3.3)$$

This fixes every factorial and density version used below.

Proposition 4 (Attained density formula, Proposition A). *For an actual $(n+2)$ -facet polytope,*

$$\frac{\mathcal{P}_n(K)}{c_n} = \max_{t \in I} \frac{D(a)}{n+1} f_{Y_t}(0). \quad (3.4)$$

The maximum is attained at $t^* = \langle a, \lambda^* \rangle \in I$.

Proof. Since $R = \{\mathbf{1}, a\}^\perp$, $\lambda + R = \{s \in H : \langle a, s \rangle = \langle a, \lambda \rangle\}$. Every real level is represented by a $\lambda \in H$. Lemma 2 gives a global maximum at λ^* , whose strictly positive weights put t^* strictly between the extreme knots. Thus the maximum over the open interval I equals the global maximum; no assumption that boundary sections vanish is needed. Multiplying (2.6), Lemma 3 and (3.3), with $m-2 = n$, gives

$$\mathcal{P}_n(K) = \frac{n^n}{n!} D(a) f_{Y_{t^*}}(0).$$

Division by c_n proves the assertion. $\square$

4 The normal form and distinct actual facets

Fix $t \in I$ and select one index of each extreme value of a . Set

$$\mu = \frac{2}{\max a - \min a}, \quad \alpha = \mu(t - \min a), \quad \beta = \mu(\max a - t) = 2 - \alpha.$$

Both α, β are positive. The *centered coefficient vector* $c = a - t\mathbf{1}$ is replaced by μc ; we are not holding a second copy of t fixed in a redefined Y_t . Its extreme coordinates are $-\alpha$ and β , and its remaining n coordinates form $b \in [-\alpha, \beta]^n$. Pairwise distances are unchanged by centering; positive scaling multiplies D by μ and divides a density at zero by μ .

For $U = \beta E' - \alpha E''$ direct convolution yields

$$g(u) = \frac{1}{\alpha + \beta} \begin{cases} e^{-u/\beta}, & u \geq 0, \\ e^{u/\alpha}, & u \leq 0. \end{cases} \quad (4.1)$$

For example, at $u > 0$ integrate $\beta^{-1} e^{-(u+\alpha r)/\beta} e^{-r}$ over $r \geq 0$. The expressions meet continuously at zero. Independence gives $f_{\mu Y_t}(0) = \mathbb{E}g(-W_b)$, where $W_b = \sum b_j E_j$. The two outer coordinates contribute 2 to D , and each middle coordinate contributes $|\beta - b_j| + |b_j + \alpha| = 2$. Consequently

$$\frac{D(a) f_{Y_t}(0)}{n+1} = \Psi(\alpha, b) := \left(1 + \frac{D(b)}{2(n+1)}\right) \mathbb{E} \exp \left(-\frac{W_b^+}{\alpha} - \frac{W_b^-}{\beta} \right). \quad (4.2)$$

In particular the positive part is divided by α , since the convolution evaluates g at $-W_b$.

Lemma 5. *For an actual $(n + 2)$ -facet polytope, the middle vector b above is nonzero for every $t \in I$.*

Proof. If $b = 0$, then $v = a - t\mathbf{1} \in \ker W$ is supported on the two chosen extreme indices, with $v_+ > 0 > v_-$. Thus $v_+w_+ + v_-w_- = 0$ implies $w_+ = (-v_-/v_+)w_-$. The areas are positive, so the outer unit normals coincide. But the exposed face in a specified direction is unique; two distinct actual facets cannot have the same outer unit normal. This contradiction also covers repeated extreme knots: any unselected nonzero extreme would already have made $b \neq 0$. $\square$

5 Conditional log-concavity and the Laplace estimate

For $b \neq 0$ in $\mathbb{R}^n$ write

$$p = \mathbb{P}(W_b > 0), \quad P = \mathbb{E}W_b^+, \quad N = \mathbb{E}W_b^-.$$

Each nonzero signed exponential summand has a log-concave density on its half-line, extended by zero. Zero summands are omitted. Prékopa's marginal theorem [4, Theorems 6–8], applied to $f(x - y)g(y)$, shows that their convolution is log-concave. At least one summand has a density, so W_b is absolutely continuous and $\mathbb{P}(W_b = 0) = 0$. Also $\mathbb{E}|W_b| \leq \sum |b_j| < \infty$. When both signs occur the law has positive probability on each half-line (take the positive summands large and negative ones small, or reverse these choices). Thus $0 < p < 1$ exactly in the mixed case; $p = 1$ or 0 in the positive-only or negative-only case.

Restriction to a half-line, reflection and positive normalization preserve log-concavity. Hence, when defined, $X_+ = (W_b \mid W_b > 0)$ and $X_- = (-W_b \mid W_b < 0)$ have log-concave densities on $(0, \infty)$, no atom at zero, and finite strictly positive means. The same holds for the unconditioned positive variable in a one-sided case.

Lemma 6 (Classical HNBUE–Laplace comparison). *If $X \geq 0$ has a log-concave probability density, no atom at zero, and finite mean $0 < \eta < \infty$, then for $s \geq 0$,*

$$\mathbb{E}e^{-sX} \leq \frac{1}{1 + s\eta}. \quad (5.1)$$

Proof. Let $H(x) = \mathbb{P}(X > x)$ and $G(x) = \int_x^\infty H(r) dr$, for $x \geq 0$. The survival function is log-concave: apply Prékopa's marginal theorem to $f(x + r)\mathbf{1}_{r \geq 0}$. Its finite marginal is allowed to have a convex half-line support; smoothness and strict log-concavity are not required. We have continuous H , $H(0) = 1$, $G(0) = \eta$, and $G' = -H$. Concavity of $\log H$ makes $H(x + r)/H(x)$ nonincreasing in x where $H(x) > 0$, for fixed $r \geq 0$. Therefore

$$\frac{G(x)}{H(x)} = \int_0^\infty \frac{H(x + r)}{H(x)} dr \leq \int_0^\infty H(r) dr = \eta.$$

If $H(x) = 0$, then $G(x) = 0$ too. It follows that $G' \leq -G/\eta$ and $G(x) \leq \eta e^{-x/\eta}$, the HNBUE bound. On $[0, R]$, integration by parts gives the exact identity

$$\int_0^R e^{-sx} H(x) dx = \eta - e^{-sR} G(R) - s \int_0^R e^{-sx} G(x) dx.$$

The boundary term tends to zero, since $G(R) \rightarrow 0$; the exponential bound also ensures integrability. Thus the left side at infinity is at least $\eta - s\eta/(s + 1/\eta) = \eta/(1 + s\eta)$. Tonelli applied to $1 - e^{-sX} = \int_0^X s e^{-sx} dx$ now yields

$$\mathbb{E}e^{-sX} = 1 - s \int_0^\infty e^{-sx} H(x) dx \leq 1 - \frac{s\eta}{1 + s\eta}.$$

At $s = 0$ the assertion is equality. Finite support endpoints cause no extra term, since both the survival function and its tail integral then vanish. $\square$

6 The universal two-bracket reduction

The following is the original deterministic reduction, together with its probability comparison. Its proof does not require the box constraint.

Proposition 7 (Proposition B). *Let $n \geq 1$, $\alpha, \beta > 0$, $\alpha + \beta = 2$, and $b \neq 0$. Assume*

$$D(b)p^2 \leq (n+1)P, \quad D(b)(1-p)^2 \leq (n+1)N, \quad (6.1)$$

and that the first inequality is strict when $0 < p < 1$. Then $\Psi(\alpha, b) < 1$.

Proof. Put $d = D(b)/(2(n+1)) \geq 0$. If $p = 1$, then $P > 0$ and $d \leq P/2 < P/\alpha$, since $\alpha < 2$. Lemma 6 gives $\Psi \leq (1+d)/(1+P/\alpha) < 1$. If $p = 0$, use $N > 0$ and $d \leq N/2 < N/\beta$ instead. These cases include $d = 0$.

For $0 < p < 1$ set $A = P/p > 0$, $B = N/(1-p) > 0$. Conditioning on the two half-lines and applying Lemma 6 gives

$$\mathbb{E}e^{-W_b^+/\alpha - W_b^-/\beta} \leq \frac{p\alpha}{\alpha + A} + \frac{(1-p)\beta}{\beta + B}. \quad (6.2)$$

Let Q be one minus $(1+d)$ times the right side. Direct expansion gives

$$\begin{aligned} \frac{(\alpha + A)(\beta + B)}{\alpha\beta}Q &= \left[\frac{P}{\alpha} + \frac{N}{\beta} - d \right] \\ &\quad + \frac{AB}{\alpha\beta} \left[1 - d \left(\frac{p^2\alpha}{P} + \frac{(1-p)^2\beta}{N} \right) \right]. \end{aligned} \quad (6.3)$$

For completeness, before dividing by $\alpha\beta$ the numerator is

$$pA\beta + (1-p)B\alpha - d\alpha\beta + AB - d[p\alpha B + (1-p)\beta A].$$

Substitution of $P = pA$ and $N = (1-p)B$ gives (6.3), with no use of a numerical test. The two assumptions say $2dp^2 \leq P$ and $2d(1-p)^2 \leq N$. The second bracket is nonnegative because its subtracted terms are at most $\alpha/2$ and $\beta/2$; it is strictly positive because the first bound is strict and $\alpha > 0$. This remains valid at $d = 0$. For the first bracket,

$$\sqrt{2d} \leq \sqrt{P} + \sqrt{N}, \quad (\sqrt{P} + \sqrt{N})^2 \leq (\alpha + \beta)(P/\alpha + N/\beta) = 2(P/\alpha + N/\beta).$$

The first inequality adds the two square-root consequences of (6.1); the second is weighted Cauchy–Schwarz. Thus that bracket is nonnegative. All prefactors and denominators are positive, so $Q > 0$. Equation (6.2) proves the result. $\square$

Reviewed formal support at this use site. The unchanged algebra above is supported by T088P133.rational_mixture_lt_one [12, 13], with $q = 1-p$, $A = P/p$, $B = N/q$ and the moments in (6.1). The independent audit’s T088P133Review.original_normalized_slack checks exactly (6.3); scalar_majorization_use composes the strict bound with an *assumed scalar* inequality of the form (6.2). It does not prove (6.2) for an expectation. The reduced $p = 1$ and $p = 0$ expressions are covered separately. Appendix C gives the precise hypotheses, declaration mapping

and trust boundary. The probability comparison, certificate and geometric assembly remain the mathematical arguments in this paper.

Under scaling $b \mapsto \lambda b$, $\lambda > 0$, the quantities P, N, d, A, B scale by λ , while p does not. The first bracket scales linearly and the second remains constant. The cleared deficit, divided by λ , is affine in λ . This explains the separation but is not a necessity claim about the geometric inequality.

7 Partial fractions and the exact four-variable estimate

We prove (1.2) with a fully explicit finite certificate.

Lemma 8 (Signed partial fractions). *For distinct nonzero real $c_1, \dots, c_r$, set $A_j = \prod_{i \neq j} c_j / (c_j - c_i)$ and $W = \sum_j c_j E_j$. Then*

$$\mathbb{P}(W > 0) = \sum_{c_j > 0} A_j, \quad \mathbb{E}W^+ = \sum_{c_j > 0} A_j c_j. \quad (7.1)$$

Proof. The integrable signed density $g(x) = \sum_j A_j |c_j|^{-1} e^{-x/c_j} \mathbf{1}_{x c_j > 0}$ has bilateral Laplace transform $\sum_j A_j / (1 + s c_j)$ in the strip $|\Re s| < \min_j 1/|c_j|$. The same formula for a negative c_j follows by putting $x = -r$ in the integral on the negative half-line. Multiplying the product $\prod_i (1 + s c_i)^{-1}$ by $1 + s c_j$ at $s = -1/c_j$ gives exactly A_j . The partial fractions have the same simple poles; their difference has none and vanishes at infinity, so the transforms agree. Both signed measures have exponential moments in a smaller strip. Their transforms are holomorphic there, so equality on a real interval extends to the imaginary axis. Fourier uniqueness for finite signed measures identifies the laws. For example, that uniqueness follows by convolution with Gaussian approximate identities, Fourier inversion for each convolution and weak convergence back to the measure. Integrating the positive-side density and its first moment gives (7.1). $\square$

Proposition 9 (Proposition C). *For every $b \in \mathbb{R}^4$, $D(b) \mathbb{P}(W_b > 0)^2 \leq 5 \mathbb{E}W_b^+$. The inequality is strict whenever b has a positive entry, in particular whenever it has entries of both signs.*

Proof. If b has no positive entry, $p = P = 0$, including $b = 0$. Otherwise, by permutation write its entries as k positives $u_1 \geq \dots \geq u_k > 0$, l negatives $-v_1 \leq \dots \leq -v_l < 0$, and $z = 4 - k - l$ zeros. Use ordered gaps

$$u_i = \sum_{a=i}^k x_a, \quad v_j = \sum_{b=j}^l y_b, \quad x_a, y_b \geq 0, \quad x_k > 0, \quad (7.2)$$

and impose $y_l > 0$ only when $l > 0$. If $l = 0$ there are no y variables or negative endpoints and all products over that list are 1. Set

$$\mathcal{D} = \prod_{i=1}^k \prod_{j=1}^l (u_i + v_j) > 0, \quad \mathcal{V} = \prod_{a < b} (u_a - u_b).$$

For $e = 0, 1$ define the alternating polynomial

$$\tilde{N}_e = \sum_{i=1}^k (-1)^{i-1} u_i^{k-1+l+e} \mathcal{V}(u_1, \dots, \hat{u}_i, \dots, u_k) \prod_{a \neq i} \prod_{j=1}^l (u_a + v_j). \quad (7.3)$$

The hat means omission. Each alternating polynomial vanishes when $u_a = u_b$, hence is divisible by $u_a - u_b$. These distinct linear factors are relatively prime in $\mathbb{Q}[u, v]$, so their product $\mathcal{V}$ divides it.

Let $N_p = \tilde{N}_0/\mathcal{V}$ and $N_P = \tilde{N}_1/\mathcal{V}$, as polynomials, and then substitute (7.2). On the dense locus of all positive gaps, (7.1) gives

$$p = N_p/\mathcal{D}, \quad P = N_P/\mathcal{D}. \quad (7.4)$$

Indeed zero coordinates are omitted from the pole list, and

$$\mathcal{V} = (-1)^{i-1} \prod_{a \neq i} (u_i - u_a) \mathcal{V}(u_1, \dots, \hat{u}_i, \dots, u_k),$$

which fixes both the exponent $k - 1 + l$ and its alternating sign.

These identities extend to all the closed gap strata with positive surviving endpoints. To see this without a genericity assumption, couple $W_{b(r)}$ and W_b using the same exponentials when $b^{(r)} \rightarrow b \neq 0$. They converge almost surely; the no-atom property at zero gives convergence of the sign indicators almost surely and hence of p . In a bounded coefficient neighborhood, $|W_{b(r)}| \leq C \sum E_i$, so dominated convergence gives continuity of P . The polynomial quotients in (7.4) are continuous because $\mathcal{D} > 0$, even for repeated same-sign knots. This proves (7.4) on the whole stratum.

If $s_1 \leq \dots \leq s_4$ are the sorted entries of b , then $D(b) = \sum_{i=1}^4 (2i - 5)s_i$: s_i occurs positively $i - 1$ times and negatively $4 - i$ times in the sum of differences. Thus D is linear in the gaps, with zeros still counted in the ranks. We have the exact identity

$$5P - D(b)p^2 = \frac{T_{klz}}{\mathcal{D}^2}, \quad T_{klz} = 5N_P\mathcal{D} - DN_p^2. \quad (7.5)$$

Appendix A lists every nonzero coefficient of each of the ten possible T_{klz} and derives a division-free recurrence that also reconstructs them. All listed coefficients are positive and all omitted coefficients are zero. Each polynomial contains

$$k(k+1)x_k^{2kl+1} \quad (7.6)$$

with positive coefficient. Since $x_k > 0$, this term survives every zero interior gap, regardless of the negative gaps. Thus $T_{klz} > 0$ on its entire stratum. This proves the proposition.

If an endpoint tends to zero, reclassify the limiting vector by its actual positive, negative and zero entries; its own listed certificate applies. Strictness is supplied by its own endpoint monomial, not inferred from continuity. No continuity at $b = 0$ is asserted or needed. $\square$

8 Minimum facets, affine invariance and equality

A full-dimensional n -polytope has at least $n + 1$ facets. Indeed, after putting zero in its interior, write its irredundant inequalities as $\langle q_i, x \rangle \leq 1$. Its polar is $\text{conv}(q_i)$: the inclusion in the polar is immediate, and separation gives the reverse using boundedness and the interior point. Irredundancy makes each q_i a vertex (a nonvertex inequality would follow from the others). The polar is full-dimensional and needs at least $n + 1$ vertices. With exactly $n + 1$ vertices they are affinely independent, so it is a simplex. Its polar is also a simplex: intersecting every choice of n of its facet hyperplanes gives the $n + 1$ dual vertices, as follows directly from affine barycentric coordinates and the strictly positive barycentric coordinates of zero. Thus K with $n + 1$ facets is a simplex.

For an invertible linear map A , the transformed facet normal is $A^{-T}u_i/|A^{-T}u_i|$ and its area is $|\det A| |A^{-T}u_i| f_i$. For example, map a unit-height prism over the facet: its volume scales by $|\det A|$, while its new height is $1/|A^{-T}u_i|$. Hence area vectors transform by $|\det A| A^{-T}$. Equation (2.2) proves

$$\Pi(AK) = |\det A| A^{-T} \Pi K.$$

The absolute determinant of this map is $|\det A|^{n-1}$, which is also the factor in $V_n(AK)^{n-1}$. Translations preserve projection volumes. Thus $\mathcal{P}_n$ is affine invariant, including orientation reversals, agreeing with Source P, `lem:affine`.

The standard simplex $T_n = \text{conv}(0, e_1, \dots, e_n)$ has volume $1/n!$. Its area vectors are $-e_i/(n-1)!$ and $(e_1 + \dots + e_n)/(n-1)!$. For the last facet this follows by projecting onto a coordinate hyperplane, with projection Jacobian $1/\sqrt{n}$. Every one of its $n+1$ full-size minors has absolute determinant $1/((n-1)!)^n$. Therefore

$$V_n(\Pi T_n) = \frac{n+1}{((n-1)!)^n}, \quad \mathcal{P}_n(T_n) = \frac{(n+1)n^n}{n!} = c_n. \quad (8.1)$$

Affine invariance proves this for every simplex; this is the interface of Source P, `prop:simplex`, with its constants shown explicitly.

Proof of Theorem 1. The only facet counts are five and six. With five, K is a simplex, and (8.1) gives $c_4 = 5 \cdot 4^4/4! = 160/3$. With six, Proposition 4 evaluates $\mathcal{P}_4(K)/c_4$ at an attained $t^* \in I$. Equation (4.2) gives $\Psi(\alpha, b)$ there, with $\alpha, \beta > 0$ and $\alpha + \beta = 2$. Lemma 5 gives $b \neq 0$. Proposition 9 applies to b and $-b$, and is strict in the mixed case. These are precisely the hypotheses of Proposition 7, so $\mathcal{P}_4(K)/c_4 < 1$. Attainment is essential: pointwise strictness would not by itself exclude a supremum equal to one. The exhibited maximum supplies it. No simple-polytope or positive-interior-gap assumption has been used. $\square$

9 Limits of the auxiliary method

A negative coefficient in T rejects that nonnegative-coefficient certificate; it does not refute the underlying inequality. There is a separate exact reason the auxiliary inequality $D(b)p^2 \leq (n+1)P$ fails for $n \geq 5$. Put $r = n-1$, $b = (-1, \varepsilon, \dots, \varepsilon)$, with $\varepsilon > 0$, and write $W = \varepsilon A - Z$, where $A \sim \text{Gamma}(r, 1)$ and $Z \sim \text{Exp}(1)$ are independent. Conditional on A , direct integration gives

$$\mathbb{P}(Z < \varepsilon A) = 1 - e^{-\varepsilon A}, \quad \mathbb{E}[(\varepsilon A - Z)^+ | A] = \varepsilon A - 1 + e^{-\varepsilon A}.$$

Since $\mathbb{E}e^{-\varepsilon A} = (1 + \varepsilon)^{-r}$,

$$p = 1 - (1 + \varepsilon)^{-r}, \quad P = r\varepsilon - 1 + (1 + \varepsilon)^{-r}, \quad D = r(1 + \varepsilon). \quad (9.1)$$

Taylor expansion at zero gives $p = r\varepsilon + O(\varepsilon^2)$, $P = r(r+1)\varepsilon^2/2 + O(\varepsilon^3)$, and hence

$$\lim_{\varepsilon \downarrow 0} \frac{Dp^2}{(n+1)P} = \frac{2(n-1)^2}{n(n+1)} > 1 \quad (n \geq 5).$$

For the last inequality, the numerator minus denominator is $n^2 - 5n + 2$, positive at 5 and increasing thereafter. Thus small positive ε supplies actual auxiliary failures. This concerns the separated sufficient condition only. It has no implication that the geometric inequality fails in dimensions 5–8. The unseparated normal form with its geometric box constraint remains unproved in those dimensions in this work. We make no claim of an optimal auxiliary constant 5, no positive all- n threshold theorem, and no result for four-polytopes with at least seven facets.

A Complete exact certificate and reconstruction

This appendix contains the full sparse polynomials in (7.5), not sampled values. Together with (7.3) it is a finite explicit certificate. The original author algorithm computes the two alternating numerators, divides exactly by each Vandermonde factor in $\mathbb{Q}[x, y]$, forms T , and checks coefficients. Its sparse arithmetic stores exponent tuples and exact rational coefficients; addition combines equal tuples, multiplication convolves exponents, and exact division cancels lexicographic leading terms until the remainder is zero. Nondivisibility is an error, not a discarded case. The divisibility proof in Section 7 justifies this operation, independently of its execution.

A.1 A later independent reconstruction

The following recurrence was supplied by the independent correctness review, round 3 [11], as verification of the existing operation. It is credited as a later reconstruction, not the original partial-fraction proof and not a changed author theorem. For convenience here use zero-based indices $0 \leq i < k$, $0 \leq j < l$. Let U_i and V_j be the completion times of the remaining successive independent exponential phases with means $u_i, \dots, u_{k-1}$ and $v_j, \dots, v_{l-1}$. Put $q_{ij} = \mathbb{P}(U_i > V_j)$ and $R_{ij} = \mathbb{E}(U_i - V_j)^+$. The first U phase wins the race with probability $v_j/(u_i + v_j)$; the first V phase wins with probability $u_i/(u_i + v_j)$. These follow by integrating the two exponential densities. Conditional on either outcome, memorylessness leaves the unfinished phase independent with its original exponential law. Elapsed time subtracts from both completion times and cancels in their difference. Thus both q and R obey the same two-branch recurrence. The boundary at $i = k$ is $(q, R) = (0, 0)$; at $j = l$ and $i < k$ it is $(1, \sum_{a=i}^{k-1} u_a)$. The state (k, l) is unreachable from a live race; take the $i = k$ boundary first.

Clear the rectangle denominator $\mathcal{D}_{ij} = \prod_{a=i}^{k-1} \prod_{b=j}^{l-1} (u_a + v_b)$. For either numerator N the resulting polynomial recurrence is

$$\begin{aligned} N_{ij} = & v_j \prod_{b=j+1}^{l-1} (u_i + v_b) N_{i+1,j} \\ & + u_i \prod_{a=i+1}^{k-1} (u_a + v_j) N_{i,j+1}. \end{aligned} \tag{A.1}$$

The boundary numerators equal the boundary values, with denominator one. This terminates as $i+j$ increases, uses only integer polynomial arithmetic, and is valid even for repeated positive means. At $(0, 0)$ it gives N_p, N_P . On the distinct locus these equal the original partial-fraction quotients; polynomial identity then makes the two constructions identical everywhere. Formula (A.1) provides a second explicit way to verify every table below, with no interpolation or numerical probability computation.

The archive retains the original author script `author-certificate.py`. It also retains the round-3 script `independent_certificate.py` and its output `certificate-witness.json`. A new exact driver compares them. The JSON records all five polynomials $N_p, N_P, \mathcal{D}, D, T$, not merely T . Original notices are retained. The full sparse T expansions below were generated only after coefficient-for-coefficient comparison of the independent recurrence, the author algorithm and the retained witness. Reproduction commands and exact byte hashes accompany this manuscript. The original geometry code is a floating diagnostic only; its outputs do not prove any step above.

A.2 Sparse coefficient notation and strict terms

In each table, $(e; c)$ denotes the term $c \prod_{i=1}^{k+l} z_i^{e_i}$ with the variable order

$$z = (x_1, \dots, x_k, y_1, \dots, y_l).$$

Every nonzero term is listed exactly once; omitted terms have coefficient zero. All entries are positive integers. The following summary also records the pure endpoint term, which stays positive even when all interior gaps vanish.

(k, l, z)	terms	degree	least coefficient	endpoint term
$(1, 0, 3)$	1	1	2	$2x_1$
$(1, 1, 2)$	2	3	2	$2x_1^3$
$(1, 2, 1)$	5	5	2	$2x_1^5$
$(1, 3, 0)$	14	7	2	$2x_1^7$
$(2, 0, 2)$	2	1	2	$6x_2$
$(2, 1, 1)$	15	5	2	$6x_2^5$
$(2, 2, 0)$	118	9	2	$6x_2^9$
$(3, 0, 1)$	3	1	2	$12x_3$
$(3, 1, 0)$	84	7	2	$12x_3^7$
$(4, 0, 0)$	4	1	2	$20x_4$

A.3 Pattern (1,0,3)

Variable order: (x_1) .

e	c
(1)	2

A.4 Pattern (1,1,2)

Variable order: (x_1, y_1) .

e	c	e	c
(2, 1)	2	(3, 0)	2

A.5 Pattern (1,2,1)

Variable order: (x_1, y_1, y_2) .

e	c	e	c	e	c
(3, 0, 2)	5	(4, 0, 1)	6	(5, 0, 0)	2
(3, 1, 1)	5	(4, 1, 0)	2		

A.6 Pattern (1,3,0)

Variable order: (x_1, y_1, y_2, y_3) .

e	c	e	c	e	c
(4, 0, 0, 3)	5	(5, 0, 0, 2)	15	(6, 0, 0, 1)	12
(4, 0, 1, 2)	10	(5, 0, 1, 1)	20	(6, 0, 1, 0)	6
(4, 0, 2, 1)	5	(5, 0, 2, 0)	5	(6, 1, 0, 0)	2
(4, 1, 0, 2)	5	(5, 1, 0, 1)	10	(7, 0, 0, 0)	2
(4, 1, 1, 1)	5	(5, 1, 1, 0)	5		

A.7 Pattern (2,0,2)

Variable order: (x_1, x_2) .

e	c	e	c
(0, 1)	6	(1, 0)	2

A.8 Pattern (2,1,1)

Variable order: (x_1, x_2, y_1) .

e	c	e	c	e	c
(0, 2, 3)	3	(1, 2, 2)	14	(2, 2, 1)	16
(0, 3, 2)	12	(1, 3, 1)	28	(2, 3, 0)	10
(0, 4, 1)	16	(1, 4, 0)	14	(3, 0, 2)	2
(0, 5, 0)	6	(2, 0, 3)	2	(3, 1, 1)	4
(1, 1, 3)	3	(2, 1, 2)	8	(3, 2, 0)	2

A.9 Pattern (2,2,0)

Variable order: (x_1, x_2, y_1, y_2) .

e	c	e	c	e	c
(0, 3, 0, 6)	20	(1, 6, 2, 0)	38	(3, 3, 2, 1)	102
(0, 3, 1, 5)	60	(1, 7, 0, 1)	128	(3, 3, 3, 0)	7
(0, 3, 2, 4)	60	(1, 7, 1, 0)	60	(3, 4, 0, 2)	218
(0, 3, 3, 3)	20	(1, 8, 0, 0)	26	(3, 4, 1, 1)	198
(0, 4, 0, 5)	74	(2, 1, 0, 6)	20	(3, 4, 2, 0)	32
(0, 4, 1, 4)	176	(2, 1, 1, 5)	60	(3, 5, 0, 1)	136
(0, 4, 2, 3)	130	(2, 1, 2, 4)	60	(3, 5, 1, 0)	64
(0, 4, 3, 2)	28	(2, 1, 3, 3)	20	(3, 6, 0, 0)	36
(0, 5, 0, 4)	118	(2, 2, 0, 5)	120	(4, 0, 0, 5)	6
(0, 5, 1, 3)	212	(2, 2, 1, 4)	285	(4, 0, 1, 4)	14
(0, 5, 2, 2)	108	(2, 2, 2, 3)	210	(4, 0, 2, 3)	10
(0, 5, 3, 1)	14	(2, 2, 3, 2)	45	(4, 0, 3, 2)	2
(0, 6, 0, 3)	116	(2, 3, 0, 4)	282	(4, 1, 0, 4)	32
(0, 6, 1, 2)	152	(2, 3, 1, 3)	508	(4, 1, 1, 3)	60
(0, 6, 2, 1)	50	(2, 3, 2, 2)	262	(4, 1, 2, 2)	32
(0, 6, 3, 0)	3	(2, 3, 3, 1)	36	(4, 1, 3, 1)	4

Pattern (2,2,0), continued

e	c	e	c	e	c
(0, 7, 0, 2)	80	(2, 4, 0, 3)	368	(4, 2, 0, 3)	68
(0, 7, 1, 1)	72	(2, 4, 1, 2)	480	(4, 2, 1, 2)	96
(0, 7, 2, 0)	12	(2, 4, 2, 1)	160	(4, 2, 2, 1)	34
(0, 8, 0, 1)	34	(2, 4, 3, 0)	11	(4, 2, 3, 0)	2
(0, 8, 1, 0)	16	(2, 5, 0, 2)	330	(4, 3, 0, 2)	78
(0, 9, 0, 0)	6	(2, 5, 1, 1)	294	(4, 3, 1, 1)	74
(1, 2, 0, 6)	30	(2, 5, 2, 0)	48	(4, 3, 2, 0)	12
(1, 2, 1, 5)	90	(2, 6, 0, 1)	188	(4, 4, 0, 1)	50
(1, 2, 2, 4)	90	(2, 6, 1, 0)	88	(4, 4, 1, 0)	24
(1, 2, 3, 3)	30	(2, 7, 0, 0)	44	(4, 5, 0, 0)	14
(1, 3, 0, 5)	148	(3, 0, 0, 6)	5	(5, 0, 0, 4)	2
(1, 3, 1, 4)	352	(3, 0, 1, 5)	15	(5, 0, 1, 3)	4
(1, 3, 2, 3)	260	(3, 0, 2, 4)	15	(5, 0, 2, 2)	2
(1, 3, 3, 2)	56	(3, 0, 3, 3)	5	(5, 1, 0, 3)	8
(1, 4, 0, 4)	286	(3, 1, 0, 5)	46	(5, 1, 1, 2)	12
(1, 4, 1, 3)	512	(3, 1, 1, 4)	109	(5, 1, 2, 1)	4
(1, 4, 2, 2)	261	(3, 1, 2, 3)	80	(5, 2, 0, 2)	12
(1, 4, 3, 1)	35	(3, 1, 3, 2)	17	(5, 2, 1, 1)	12
(1, 5, 0, 3)	324	(3, 2, 0, 4)	140	(5, 2, 2, 0)	2
(1, 5, 1, 2)	420	(3, 2, 1, 3)	256	(5, 3, 0, 1)	8
(1, 5, 2, 1)	138	(3, 2, 2, 2)	135	(5, 3, 1, 0)	4
(1, 5, 3, 0)	9	(3, 2, 3, 1)	19	(5, 4, 0, 0)	2
(1, 6, 0, 2)	258	(3, 3, 0, 3)	220		
(1, 6, 1, 1)	230	(3, 3, 1, 2)	296		

A.10 Pattern (3,0,1)Variable order: (x_1, x_2, x_3) .

e	c	e	c	e	c
(0, 0, 1)	12	(0, 1, 0)	6	(1, 0, 0)	2

A.11 Pattern (3,1,0)Variable order: (x_1, x_2, x_3, y_1) .

e	c	e	c	e	c
(0, 0, 2, 5)	3	(0, 5, 1, 1)	12	(2, 0, 2, 3)	46
(0, 0, 3, 4)	49	(0, 5, 2, 0)	6	(2, 0, 3, 2)	80
(0, 0, 4, 3)	126	(1, 0, 1, 5)	2	(2, 0, 4, 1)	60
(0, 0, 5, 2)	132	(1, 0, 2, 4)	31	(2, 0, 5, 0)	16
(0, 0, 6, 1)	64	(1, 0, 3, 3)	132	(2, 1, 0, 4)	8
(0, 0, 7, 0)	12	(1, 0, 4, 2)	190	(2, 1, 1, 3)	54
(0, 1, 1, 5)	4	(1, 0, 5, 1)	116	(2, 1, 2, 2)	132
(0, 1, 2, 4)	80	(1, 0, 6, 0)	26	(2, 1, 3, 1)	128
(0, 1, 3, 3)	300	(1, 1, 0, 5)	3	(2, 1, 4, 0)	42
(0, 1, 4, 2)	410	(1, 1, 1, 4)	37	(2, 2, 0, 3)	16
(0, 1, 5, 1)	244	(1, 1, 2, 3)	226	(2, 2, 1, 2)	68
(0, 1, 6, 0)	54	(1, 1, 3, 2)	440	(2, 2, 2, 1)	88

Pattern (3,1,0), continued					
e	c	e	c	e	c
(0, 2, 0, 5)	3	(1, 1, 4, 1)	340	(2, 2, 3, 0)	36
(0, 2, 1, 4)	49	(1, 1, 5, 0)	92	(2, 3, 0, 2)	10
(0, 2, 2, 3)	262	(1, 2, 0, 4)	14	(2, 3, 1, 1)	20
(0, 2, 3, 2)	480	(1, 2, 1, 3)	134	(2, 3, 2, 0)	10
(0, 2, 4, 1)	360	(1, 2, 2, 2)	360	(3, 0, 0, 4)	2
(0, 2, 5, 0)	96	(1, 2, 3, 1)	360	(3, 0, 1, 3)	8
(0, 3, 0, 4)	12	(1, 2, 4, 0)	120	(3, 0, 2, 2)	12
(0, 3, 1, 3)	104	(1, 3, 0, 3)	28	(3, 0, 3, 1)	8
(0, 3, 2, 2)	264	(1, 3, 1, 2)	124	(3, 0, 4, 0)	2
(0, 3, 3, 1)	256	(1, 3, 2, 1)	164	(3, 1, 0, 3)	4
(0, 3, 4, 0)	84	(1, 3, 3, 0)	68	(3, 1, 1, 2)	12
(0, 4, 0, 3)	16	(1, 4, 0, 2)	14	(3, 1, 2, 1)	12
(0, 4, 1, 2)	68	(1, 4, 1, 1)	28	(3, 1, 3, 0)	4
(0, 4, 2, 1)	88	(1, 4, 2, 0)	14	(3, 2, 0, 2)	2
(0, 4, 3, 0)	36	(2, 0, 0, 5)	2	(3, 2, 1, 1)	4
(0, 5, 0, 2)	6	(2, 0, 1, 4)	12	(3, 2, 2, 0)	2

A.12 Pattern (4,0,0)

Variable order: (x_1, x_2, x_3, x_4) .

e	c	e	c
(0, 0, 0, 1)	20	(0, 1, 0, 0)	6
(0, 0, 1, 0)	12	(1, 0, 0, 0)	2

B Positioning, provenance and coverage

Positioning requirements (PQ1–PQ5). PQ1 is the exact bounded class in Theorem 1, with every six-facet case strict. PQ2 is project novelty and no antecedent found in bounded recorded searches, without worldwide priority. PQ3 credits Source P, FHLX Section 4, r132 R1–R4/($\star$) and the classical Prékopa and HNBUE/Laplace literature. PQ4 separates the full exact certificate from the floating geometry diagnostic. PQ5 is the precise auxiliary failure and geometric nonconclusion of Section 9. The assessment concerns their joint assembly, not a newly invented classical ingredient or a best auxiliary constant.

Actual primary reading. For the round-1 draft, Source P’s README, complete TeX and references were read, including its actual facet, affine and simplex labels. Its bytes match the pinned tree. The FHLX v1 TeX introduction/preliminaries and all of Section 4 (lines 157–374 and 991–1602 of the retained source) were read. The FHLX PDF was byte-authenticated, not body-read or visually inspected. Prékopa 1973, Theorem 3 with its induction proof and Theorems 6–8 with proofs, were read from the actual PDF text. The needed measurable, convex-support and finite-integral hypotheses are exposed above. Other historical originals for Minkowski, Brunn, Jacobi, zonotope volume and the Gamma–Dirichlet identity were not acquired; their needed operations have direct justifications here.

Saroglou 2011 was unavailable at the author’s host in the earlier rounds because of TLS failure; the contribution reviewer obtained an archived author PDF. The round-1 draft read its open-

ing statements (Theorems 1–2 and Corollaries 1–2) and the relevant concluding discussion, not the interior proof sections. Saroglou 2015 v2’s introduction and main statements were read in its retained TeX. Klefsjö 1982/1983 was read only through the recorded Crossref metadata and abstracts; the full bodies were not obtained or read. Barlow–Proschan, Rolski and the original Curry–Schoenberg/de Boor bodies were not read. No precise unread theorem is used to replace Lemma 6. Brannen 1996/2005 and Chen–Feng–Li–Xi–Xu bodies remain unavailable in the retained record, and are not represented as read. Bibliographic mentions of them are mediated through the identified read sources. The Petty companion’s introduction was checked for scope; it supplies no operative upper-bound premise. Downloading or copying a body neither establishes reading nor grants redistribution rights.

History and chronology. The r132 report and notes were read for the exact R1–R4 and (★) antecedents; I43’s RP4 subsection, r112’s 088 paragraph and the r131 notes were checked as bounded project history. They supply no prior proof of this class in the recorded reads. The original author had a proof sketch and exact certificate before the recorded freeze at 2026-10-20 17:21:32 UTC. That freeze precedes the extended prose, not all ideas or certificate feasibility. The immutable author FINAL records sealing at 17:31:58.711847 UTC, whereas the notes describe drafting and packaging at 17:36–17:45 UTC. These time labels conflict; we preserve both and do not infer which detailed chronology is correct. The late companion and acceptance-rule reads, and the author’s disclosed temporary history lists outside its assigned root, remain disclosed. No original has been edited to repair that record.

Earlier correctness rounds ended through network and usage failures, not mathematical verdicts. Their partial code had a missing Vandermonde factor corrected before its saved output and substituted distinct-only spot samples for repeated knots. Those samples were not full proof checks. The later independent competing-phase reconstruction verifies the existing certificate operation; it is credited separately above. The parent adopted a full correctness PASS and a separate contribution SUBSTANTIAL assessment at the unchanged scope, discharging the latter’s correctness condition. Neither is acceptance, and this implementation assigns no independent review or contribution verdict.

Formal integration history. The immutable round-1 manuscript truthfully recorded formal support as pending at its time of drafting. The completed implementation and separate independent scoped review are now authenticated and integrated here, without changing either original package. The parent adopted PASS_SCOPED_SUPPORT, with no repairs, for deterministic algebra only. This is not a full mathematical, manuscript, acceptance, count or release verdict. Appendix C states its actual coverage. The parent’s earlier manuscript intake checked artifacts and sampled layout; it was not an independent all-page manuscript review.

Rights and reproducibility. Myeongjin Kang is the author, with no affiliation or supplied contact details. Original paper/prose is licensed CC BY 4.0 and original code MIT. Copied original author/reviewer code retains its actual MIT notices and is identified byte-for-byte in the support record. Third-party source, primary and historical bodies and their derivatives remain private/local under their original rights and are excluded from the manuscript payload. Build caches and tool downloads are also separate. This is a local manuscript implementation for independent review, not a release or publication. Full input bindings, commands, page-by-page visual QA, limits and detached integrity anchors are in the implementation report; integrity checks do not confer mathematical gates.

C Precisely scoped formal support

The completed Lean implementation [12] and separate independent review [13] support only the deterministic real algebra at Proposition 7. The reviewed files are supplied unchanged, with origin SHA-256 hashes and detached package anchors in the support metadata. There are 21 implementation support declarations (three definitions and 18 theorems), 20 implementation control theorems, and 15 independent semantic, type and use-site audit declarations.

Exact universal interface. For all real $\alpha, \beta, p, q, d, P, N$ satisfying

$$\begin{aligned} \alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 2, \quad 0 < p < 1, \quad q = 1 - p, \\ d \geq 0, \quad P > 0, \quad N > 0, \quad 2dp^2 < P, \quad 2dq^2 \leq N, \end{aligned} \tag{C.1}$$

T088P133.rational_mixture_lt_one proves exactly

$$(1 + d) \left[\frac{p\alpha}{\alpha + P/p} + \frac{q\beta}{\beta + N/q} \right] < 1. \tag{C.2}$$

The declaration `rational_mixture_le_one` replaces the strict first moment premise and conclusion by non-strict inequalities. The algebraic declaration `conditional_means` derives $q > 0$, $P/p > 0$, $N/q > 0$, $p(P/p) = P$ and $q(N/q) = N$; it is not a theorem about conditional expectations. The two `moment_lt_iff` and `moment_le_iff` declarations convert $2dp^2 < P$ or $\leq P$ to $2dp < P/p$ or $\leq P/p$ when $p > 0$. Denominator positivity is derived in the mixture proofs. There is no division by d , and $d = 0$ is included.

Brackets and the unchanged paper proof. Write $A = P/p$, $B = N/q$. The definitions `firstBracket`, `secondBracket` and `mixture` are respectively

$$\begin{aligned} F &= pA\beta + qB\alpha - d\alpha\beta, \\ G &= AB - d(p\alpha B + q\beta A), \\ M &= (1 + d)[p\alpha/(\alpha + A) + q\beta/(\beta + B)]. \end{aligned}$$

The declarations `firstBracket_decomposition` and `secondBracket_decomposition` prove the polynomial identities

$$\begin{aligned} F &= p\beta(A - 2dp) + q\alpha(B - 2dq) + d(2p - \alpha)^2, \\ G &= [\alpha B(A - 2dp) + \beta A(B - 2dq)]/2. \end{aligned} \tag{C.3}$$

The first uses $\alpha + \beta = 2$ and $q = 1 - p$; the second uses only the sum normalization. These are identities without sign premises. The residual hypotheses give $F \geq 0$ and $G > 0$, since $\alpha B(A - 2dp) > 0$. This is Lean’s polynomial residual/square route to the same first-bracket sign proved by Cauchy–Schwarz in the unchanged paper proof. The names `firstBracket_nonneg`, `secondBracket_nonneg` and `secondBracket_pos` record these signs. The identities `cleared_slack` and `slack_identity` give $1 - M = (F + G)/[(\alpha + A)(\beta + B)]$; the latter takes nonzero denominators. The declarations `mixture_le_one` and `mixture_lt_one` derive the corresponding bounds using positive denominators.

The independent declarations in namespace T088P133Review prove `original_first_bracket`, `original_second_bracket` and `original_normalized_slack`. They identify $F/(\alpha\beta)$ and $G/(\alpha\beta)$ with the two terms of (6.3), including the weight $AB/(\alpha\beta)$. All nonzero divisors are explicit.

The eight exact interface contracts and `definition_contract` fix the mixed wrappers, moment conversions, mean algebra, endpoint interfaces and the three definition bodies.

The audit's `scalar_majorization_use` assumes, in addition to (C.1), an arbitrary real L and the premise

$$L \leq p\alpha/(\alpha + P/p) + q\beta/(\beta + N/q).$$

It proves $(1 + d)L < 1$ by multiplication with $1 + d \geq 0$ and (C.2). It does not define L as an expectation or prove this premise for one. In the paper that premise is (6.2), proved analytically using Lemma 6.

Endpoints and controls. The implementation derives $\alpha, \beta < 2$ in `positive_sum_two_bounds`. For all real a, d, M with $0 < a < 2$, $M > 0$ and $2d \leq M$, `one_sided_lt_one` proves $(1 + d)/(1 + M/a) < 1$. It validly requires no $d \geq 0$. The declarations `p_one_bound` and `p_zero_bound` instantiate the reduced expressions in Proposition 7; they never substitute a zero probability into the mixed-case rational denominators. The independent `endpoint_expression` checks the equivalent fraction $(1 + d)a/(a + M)$, and `endpoint_negative_d` checks $d = -2$.

All 20 implementation controls have explicit elaborated types. They include the asymmetric value $12/13$, $d = 0$ value $1/3$, a non-strict equality example, and countermodels after specified assumptions are removed. The review independently checks ten exact rational configurations. Two added-premise semantic mutations compile the support but are rejected by the exact type contracts. Both historical eleven-case artifact suites passed; those are integrity tests, not proofs of algebraic meaning. These bounded controls do not assert that every hypothesis is individually necessary, and they do not construct probability or polytope realizations.

Compiler, dependencies and reproduction. All 56 advertised declarations have actual type and axiom receipts using only `propext`, `Classical.choice` and `Quot.sound`. The compiler is Lean 4.34.1. The exact commits are

```
Lean      5045d0056413266e57c625dcd7c365b10e377c52
mathlib   d13f23b723b8a846827a245b89c10fc7d3f11612
```

Eight auxiliary packages are pinned in the supplied lock. The compiler and precompiled Lean, mathlib and auxiliary dependencies are trusted. Only the supplied own support, control and audit sources are rebuilt; no clean-source dependency rebuild or complete binary-cache authentication is claimed. The new `support/formal_reproduce.py` uses relative supplied inputs, accepts explicit `--lean` and `--cache`, and builds fresh copies under the extracted manuscript's `build/work/`. It prints all actual types and axioms. Its execution was tested in a separate copy without hidden author, manuscript or formal-source roots. Metadata-only historical inventories are not execution inputs. The unchanged historical `formal/review/replay.py` still requires its companion implementation at the original project path and is not a standalone portable runner.

Entirely unformalized obligations. No Lean declaration here proves actual-facet geometry or the nonzero middle-vector exclusion; mixed-volume attainment, coarea or Gamma–Dirichlet; distributions, log-concavity, HNBUE or expectation-to-mixture majorization; the ten-pattern certificate or its probabilistic interpretation; continuity, zero/repeated-knot assembly or the derivation of a strict moment; simplex equality, every-six-facet geometric strictness or Theorem 1. The analytic and geometric proof remains necessary in full. The adopted scoped formal PASS and this integration infer no new mathematical, contribution, manuscript, acceptance, count or release gate.

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