

Removing the moment assumption from a stipulated planar no-bigeodesics theorem

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Abstract

Taking the pinned OpenAI planar no-bigeodesics theorem and its specified proof inputs as established, we give a deduction for every nonatomic law on finite nonnegative edge weights, without a moment or direction condition. A dependent circuit cap preserves exactly the set of geometric bigeodesics and supplies an integrable metric. The capped law does not have upward finite energy. We replace the required finite-energy operations by a dominated coupling of lifted limiting graphs, a transfer of labels and selected rays under underlying-coordinate modifications, and a changed-vertex barrier for independent half-plane splices. A cubic wall completes the coalescence surgery. The final budget and perturbation argument is the stipulated source's argument with status-preserving probability kernels. The additional class is precisely the laws with infinite second moment of the minimum of four independent weights. Classical contour surgery, shape theory, and the source's final counting argument are credited separately from these replacement steps.

Reproducibility. The mathematical deduction is conditional on the stipulated source identified in Section 1. The full theorem is not formally verified. The accompanying `README.md` and `build.py` give a fresh-directory PDF build; Section 8 describes the exact limited Lean support. Original prose is licensed CC BY 4.0; original supporting code is MIT licensed. No institutional affiliation is asserted.

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1 Scope, stipulated input, and attribution

Let $\mathcal{E}$ be the unoriented nearest-neighbor edges of $\mathbb{Z}^2$. For positive weights w , write $T_w(x, y)$ for the infimum of the costs of finite walks from x to y , counting every traversal. A ray is an infinite simple path whose finite segments minimize cost; a bigeodesic is such a path indexed by $\mathbb{Z}$. All weights in this paper are finite. A nonatomic nonnegative law assigns probability one to strictly positive weights simultaneously on the countable edge set.

Theorem 1.1. *Let G be any nonatomic Borel probability measure on $[0, \infty)$, and let $(t_e)_{e \in \mathcal{E}}$ be independent with common law G . Under the stipulated source convention below, almost surely there is no bigeodesic for T_t . The assertion holds simultaneously over all bi-infinite simple paths, without a prescribed direction or any moment assumption.*

Throughout, **S** denotes OpenAI’s *No bigeodesics in planar first-passage percolation*, September 24, 2026, at immutable revision `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb` of the OpenAI math repository [1]. Its theorem and its proof inputs under their stated hypotheses are stipulated mathematical inputs to this deduction. We do not audit that upstream theorem and do not apply its iid conclusion to a dependent field. We use its actual geometric, measurable-selection, and counting proof operations only after supplying the interfaces below. **AH** means Ahlberg–Hoffman, arXiv:1609.02447v2 [2]; **DH** means Damron–Hanson, arXiv:1209.3036v1 [3]. Their primary results are invoked with the hypothesis distinctions in Section 8.

S, Theorem 1.1, already covers *every* nonatomic G for which

$$\mathbb{E}[\min(X_1, X_2, X_3, X_4)^2] < \infty, \quad X_i \text{ independent with law } G. \quad (1)$$

This includes bounded laws, unbounded finite-mean laws, and some infinite-mean laws. The additional class in Theorem 1.1 is exactly the class where the expectation in (1) is infinite. We may therefore assume below that G is unbounded. No positive moment is then assumed. The full stated scope agrees with the finite-valued formulation of Damron–Hanson’s Conjecture 1 in arXiv:1512.00804v2 [4]. This comparison identifies the scope; the deduction here remains conditional on the stipulated input S.

Why an integrable surrogate is needed. The minimum moment in (1) is necessary for the usual almost-sure uniform shape theorem for the original metric. We use the account of Kesten’s necessity and Cox–Durrett/Kesten contour regularization in Cerf–Théret [6, Section 1.2]; the original Cox–Durrett and Kesten bodies have not been read for this manuscript. Thus no uniform original shape or original Busemann linearity is asserted for the additional class. The cap below has metric regularity, but fails upward finite energy (Proposition 2.4). Neither metric is an immediate black-box input to all the structural results needed by S.

What is new in this deduction. The replacement steps are the dominated lifted limiting-graph coupling (D, Section 4), label and selector transfer under the underlying-coordinate coupling (S, Section 5), the changed-vertex half-plane transfer (P, Section 6), and the short cubic wall (W, Section 4.1). The circuit cap, exact bigeodesic equivalence, shape estimates, same-status lowering, two-status density formulas, and S's budget/fork/tilt argument receive no separate novelty claim.

Open-circuit surgery and its moment-free use have classical antecedents: Auffinger–Damron–Hanson [5, Appendix C, Lemma C.5 and (26)] use open half-circuits to bypass costly geodesic portions; Cox–Durrett/Kesten contours regularize passage times as described in [6]. The direct wall antecedent is AH, Proposition 9.10, Case 1. The stationary shape theorem through Boivin is credited through AH Section 3.1 and DH's A2 discussion; Boivin's original proof is not claimed read. Section 3 retains an elementary shape derivation for completeness, not as a new result.

Logical order. First build the cap and its metric estimates. Construct coalescing reference families by the modified DH argument. Reuse AH's order/label proofs, replacing the downward step and then the upward selector step. This yields the all-ray structure. Establish the independent-splice interface, then reuse S's budget and divergent-toll contradiction with status-preserving kernels. Every geometric bigeodesic is unchanged by the cap, so the conclusion returns to the original metric. Earlier clipping, linear-cap, retention, and penalized-path screens are historical material only and are not premises of this proof.

2 The cubic circuit cap

Put $a = \text{ess inf } G$. Choose $M \geq 1$ so large that $q = G((M, \infty)) < 1/1024$. Choose $a < a_o < b_o < M$ with $G([0, a_o]) > 0$ and $G([0, b_o]) < 1/1024$. Nonatomicity at a permits this choice even for singular continuous G . Unbounded support permits $M < a_c < b_c < \infty$ with $G((M, a_c]) > 0$ and $G([b_c, \infty)) > 0$. Set

$$\begin{aligned} L_o &= [0, a_o], & H_o &= [b_o, M], & r_o &= G([0, M]), \\ L_c &= (M, a_c], & H_c &= [b_c, \infty), & r_c &= G((M, \infty)), \\ p_{a,i} &= G(L_i), & p_{b,i} &= G(H_i), & g &= \min(b_o - a_o, b_c - a_c) > 0, \\ p_* &= \min_{i \in \{o, c\}} p_{a,i}/r_i > 0. \end{aligned}$$

All four band masses are positive. Call an original edge *usable* when $t_e \in H_o \cup H_c$. Its bad probability $\epsilon < 1/512$ satisfies $8\sqrt{\epsilon} < 1$ and $6\sqrt{\epsilon} < 1/2$. These bands remain fixed.

Let V_e be independent uniform variables on $(0, 1)$, independent of t ; write $X = (t, V)$. An edge is open when $t_e \leq M$. Let L_e be the minimum length of an open simple circuit strictly enclosing both endpoints of e , and define

$$K_e = M(L_e^3 + V_e), \quad s_e = \min(t_e, K_e), \quad T = T_s. \quad (2)$$

The definition uses the minimum length only, not a choice among tied circuits.

Lemma 2.1. *Almost surely all L_e exist, $L_e \geq 10$, and $\mathbb{P}(L_e > r) \leq C \exp(-c\sqrt{r})$. The field s is a square-symmetric factor of iid, mixing under every nonzero lattice translation, has every finite moment, and has unique finite geodesics. Every open edge is unchanged.*

Proof. Failure of an open circuit in a suitable annulus of outer radius R forces a closed dual crossing of length at least cR . Erasing loops and counting at most three choices after the first edge bounds

this probability by $CR \sum_{j \geq cR} (3q)^j$. A simple circuit in the outer box has $O(R^2)$ edges. This gives the stated tail and, by countability, existence everywhere. The same argument surrounds every finite box by open circuits arbitrarily far away. A lattice circuit strictly enclosing a horizontal edge has horizontal span at least three and vertical span at least two; hence its length is at least ten. The vertical case is identical. In particular $K_e \geq 2ML_e > M$, so open edges are unchanged and $0 < s_e \leq M(L_e^3 + 1)$ has all moments.

Conditional on all open indicators, the original magnitudes have independent nonatomic conditional laws, the V_e remain independent uniforms, and the L_e are fixed. Thus the s_e are conditionally independent and nonatomic. For two distinct simple paths, condition on an edge in their symmetric difference to rule out equal sums; countability treats all finite paths. One must not freeze that edge's V_e in this no-tie argument. Existence of finite minimizers is established by the coercivity argument in Lemma 3.1. Covariance and symmetries of the minimum-length construction give the factor and mixing assertions. $\square$

Choose any measurable minimum circuit C_e for estimates, and put

$$E_x = \{e : x \text{ is strictly inside } C_e\}.$$

The tail above and the fact that x inside C_e implies $\text{dist}(x, e) \leq CL_e$ show that E_x is finite, simultaneously for every x .

Proposition 2.2 (Exact preservation). *On one event, the sets of geometric t - and s -bigeodesics are exactly equal. Every edge of each such path is unchanged. Every s -ray from a fixed root x is uncapped outside the finite set E_x , simultaneously over all rays from that root.*

Proof. Call e changed if $s_e < t_e$, so $s_e = K_e$. A bigeodesic in either metric using e must enter and leave C_e . Its segment between contacts bracketing e costs at least K_e , whereas a shorter boundary arc costs at most $ML_e/2$. Thus neither kind of bigeodesic uses a changed edge.

For the nontrivial implication, let γ be a t -bigeodesic and let π be a finite s -walk between $a, b \in \gamma$. Eliminate a changed occurrence e of π . If both endpoints lie outside or on C_e , replace the bracketed crossing by a boundary arc. If just a lies inside, replace the prefix through e by a segment of γ from a to a boundary contact and then a boundary arc to a later contact of π ; reverse the construction if just b is inside. The part of γ between its outermost contacts with C_e costs at most $ML_e/2$, by t -minimality, so the inserted portion costs at most ML_e . If both endpoints are inside, replace all of π by $\gamma[a, b]$, contained in that same outermost-contact interval and costing at most $ML_e/2$.

Every removed portion contains an edge of s -cost $K_e \geq 2ML_e$, and every inserted edge lies on the unchanged circuit or the unchanged γ . The number of changed occurrences strictly decreases, with no recursive creation of changed edges. After finitely many steps the walk is entirely unchanged and costs no more than the original s -walk. Its t -cost equals its s -cost, so t -minimality of γ proves s -minimality. Conversely an unchanged s -geodesic is t -minimal because $s \leq t$. The proof is pathwise and takes no union over an uncountable path space.

If a ray from x uses changed $e \notin E_x$, it begins outside or on C_e and eventually exits that finite region. The same boundary shortcut is a contradiction. This proves the rooted-ray statement without asserting any reattachment of a preserved tail to the same root in the other metric. $\square$

Lemma 2.3 (Finite influence and monotonicity). *On one original full event, any finite modification of the positive finite original weights, with V fixed, changes only finitely many caps. If the original weights increase, then $s' \geq s$. Under such an upward modification, an uncapped path avoiding the modified original set is unchanged.*

Proof. Tail summability gives $L_e/(1+|e|) \rightarrow 0$ uniformly over distant edges. Circuits around every finite box ensure that protecting circuits still exist after any finite modification F . For distant e , every circuit enclosing e and meeting F is longer than L_e . An old minimizer avoids F , so $L'_e \leq L_e$; a new minimizer of that length also avoids F , so it was old-admissible and $L_e \leq L'_e$. Thus equality holds far away. Raising originals only removes open circuits, giving $L'_e \geq L_e$, $K'_e \geq K_e$, and $s'_e \geq s_e$. In this upward case an uncapped edge whose original weight is not modified still has $s'_e = t_e = s_e$. $\square$

Proposition 2.4 (Failure of upward finite energy). *For unbounded G , the cap law fails DH's upward finite-energy condition and AH A2(v).*

Proof. For $f \neq e$, its status is visible from s_f : $s_f \leq M$ exactly when $t_f \leq M$. A circuit strictly enclosing both endpoints of e cannot contain e , so L_e is determined by the outside capped field s_{-e} . Consequently $s_e \leq M(L_e^3 + 1)$ conditionally on s_{-e} . The event $L_e = 10$ has positive probability and is outside-measurable. On this event the conditional probability of $s_e \geq \lambda := 1001M + 1$ is zero. Unconditionally that probability is positive: a finite closed pattern can exclude all short surrounding circuits, an open outer circuit has positive probability, and independently $t_e > \lambda$ has positive probability. Take the excluded length sufficiently large that its cap exceeds λ . Thus the positive unconditional tail is not conditionally available everywhere. $\square$

No iid law for s or restoration of its conditional finite energy is used later. Optional endpoint corrections $T_t - T_s \leq B(x) + B(y)$ with $B(x) = \sum_{e \in E_x} t_e < \infty$ do not supply uniform original shape and are not needed in the essential chain.

3 Metric and cocycle estimates

The cap meets the stationary A2 shape framework discussed in DH and AH, which credits Boivin. The following elementary derivation is included as an alternative and to state the precise uniform estimates used below.

Lemma 3.1. *Finite geodesics exist uniquely, and there is a deterministic nondegenerate square-symmetric norm μ such that, almost surely,*

$$\sup_{x,y \in Q_R} \frac{|T(x,y) - \mu(y-x)|}{R} \rightarrow 0, \quad Q_R = [-R, R]^2 \cap \mathbb{Z}^2. \quad (3)$$

For every $\eta > 0$, eventually uniformly for $x, y \in Q_R$, $T(x, y) \leq C(|x-y|_1 + \eta R)$, with C independent of η . There is a fixed L such that all geodesics between vertices of Q_{2n} have length at most Ln with probability tending to one.

Proof. Search for protecting circuits of length at most r locally, using their minimum length if present and r otherwise. The resulting cap $s_e^{(r)}$ depends on a radius $O(r)$ neighborhood, is at most $M(r^3 + 1)$, satisfies $s_e^{(r)} \leq s_e$, and agrees with s_e unless $L_e > r$. Along a coordinate segment, color edges by their position modulo a number between twice the dependence radius and Cr . Each color contains independent variables. Hoeffding and a union over the colors give, for length $\ell \geq Cr$,

$$\mathbb{P} \left(\sum_{e \text{ in segment}} s_e^{(r)} > (\mathbb{E}s_e + 1)\ell \right) \leq Cr \exp(-c\ell/r^7).$$

Take $r = \lceil \log(n+2)^4 \rceil$. A union over $O(n^3)$ coordinate segments of length at least ηn , together with $O(n^2)$ coupling failures, has summable probability

$$Cn^3 r e^{-c\eta n/r^7} + Cn^2 e^{-c\sqrt{r}}.$$

Extend shorter segments to length ηn in a fixed larger box and use positivity. Two coordinate segments give the asserted local continuity.

Choose $d \in (0, M)$ with $\mathbb{P}(t_e < d)$ so small that $\beta = 6\sqrt{\mathbb{P}(t_e < d)} < 1/2$. Since $s_e \geq \min(t_e, M)$, a simple path of length ℓ with cost below $d\ell/2$ has at least half its independent originals below d . Counting paths and bad subsets bounds the probability of any such path from $O(n^2)$ box vertices, of any length $\ell \geq n$, by $Cn^2 \sum_{\ell \geq n} \beta^\ell$. This is summable. It supplies coercivity from each fixed vertex and hence existence of finite minimizers. Coordinate routes have cost at most $C_1 n$ on the high-probability event above, so every box-to-box minimizer has length at most $\max(n, 2C_1 n/d)$. Paths leaving the box are included in the lower-cost count.

Kingman's theorem, integrability and ergodicity of each nonzero directional translation give deterministic radial limits on integer rays. Triangle inequalities and the lower-cost estimate extend them to a nondegenerate norm μ . A finite rational net and local continuity give uniform radial shape. For the shifted version fix $\delta, \eta > 0$ and choose one deterministic N so that the event

$$G_N(v) = \{|T(v, w) - \mu(w - v)| \leq \delta|w - v| \text{ for all } |w - v| \geq N\}$$

has had probability less than $c\eta^2$. The spatial ergodic theorem for this fixed event ensures fewer bad centers in Q_{2R} than the number of vertices in any ηR -ball. Every $x \in Q_R$ is therefore within ηR of a good center v . Compare $T(x, y)$ with $T(v, y)$ using local continuity and norm continuity. If $|y - v| < N$, use local continuity directly. The error is at most $C\eta R + C'\delta R + O(N)$. Countable decreasing choices of η, δ prove (3). No radius-dependent event was fed to the ergodic theorem. $\square$

Lemma 3.2 (Stationary extensions). *Let a translation-invariant extension of the cap law carry an additive covariant cocycle B with $|B(x, y)| \leq T(x, y)$. The extension need not be ergodic. There is a possibly random, globally translation-invariant linear functional ρ such that*

$$\sup_{x \in Q_R} |B(0, x) - \rho(x)|/R \longrightarrow 0.$$

Consequently for every $\varepsilon > 0$ there is finite C_ε with $|B(x, y) - \rho(y - x)| \leq \varepsilon(|x| + |y|) + C_\varepsilon$ for all x, y .

Proof. Two-sided Birkhoff on each axis gives $B(0, ke_i)/k \rightarrow \rho_i$, with the same limit in both signs. For a fixed shift w , additivity bounds the difference of the shifted and unshifted increments by $T(0, w) + T(ne_i, ne_i + w)$. Stationarity and integrability give

$$\sum_{n \geq 1} \mathbb{P}(T(ne_i, ne_i + w) > \varepsilon n) = \sum_{n \geq 1} \mathbb{P}(T(0, w) > \varepsilon n) < \infty.$$

Thus each ρ_i is invariant under every translation, by Borel–Cantelli and countability. Also $|\rho_i| \leq \mu(e_i)$.

For fixed $\alpha > 0$ let $Q_N(v)$ assert both axis errors at v are at most $\alpha|k|$ for every $|k| \geq N$. The conditional bad probability $d_N = \mathbb{E}[\mathbf{1}_{Q_N(0)^c} \mid \mathcal{I}]$ decreases to zero, where $\mathcal{I}$ is the full group-invariant sigma-field. Intersect the spatial ergodic events for all deterministic N . On each component choose a finite N with $d_N < c\eta^2$. For every $x \in Q_R$ there is $v \in Q_{\eta R}$ such that both v and $v + x_2 e_2$ are good: the two bad counts in Q_{3R} are smaller than the candidate set. This is a deterministic count

after the ergodic events have been fixed. Follow the cocycle route $0 \rightarrow v \rightarrow v + x_2 e_2 \rightarrow v + x \rightarrow x$. The middle two long axis increments have error at most $2\alpha R$; short increments contribute $O(N)$ plus an arbitrarily small linear remainder by local metric continuity. The end pieces cost at most $C(\eta + \zeta)R$. Let $R \rightarrow \infty$, then the fixed small parameters decrease to zero. Additivity gives the final extended bound. No mixing of the enlarged law was inferred from mixing of its cap marginal. $\square$

4 Reference families by a dominated lifted coupling

We first distinguish two uses of DH. Its Section 3 averaging construction, Propositions 5.1–5.2, Theorem 5.3 and Corollary 4.7 apply under A2': translation ergodicity, square symmetries, a $2 + \varepsilon$ moment, bounded norm shape, and unique passage times. Lemmas 2.1–3.1 supply these for s . DH Section 6, including Lemma 6.2 and the following triple extraction, has a *standing upward-finite-energy assumption*. We reuse the proofs of those geometric extraction steps, which use only supporting-arc geometry, stationarity and the ergodic theorem; we do not invoke their statements under an unavailable hypothesis. The later finite-energy surgery is replaced below.

Use DH's exact rounded-line targets

$$\hat{L}_\alpha = \{z \in \mathbb{Z}^2 : (z + [-1/2, 1/2]^2) \cap \{x : \rho(x) = \alpha\} \neq \emptyset\}.$$

No identification of line-target and half-plane-target graph laws is needed. Coercivity and uniqueness give finite geodesics to each such target. The DH averages, lifted to include X , s , graph and cocycle coordinates, have a stationary subsequential limit ν . Its paths are geodesics, have outdegree one, merge on intersection, and lie in the supporting arc of the limiting cocycle slope. The geometric proof of DH Lemma 6.2 and its subsequent extraction produce, on noncoalescence, a positive-probability triple of disjoint rays eventually in every translate of one of eight half-planes. Ergodicity of the lifted law is not needed: its relevant ergodic average may be random with positive expectation.

Lemma 4.1 (D: domination and selected-ray stability). *Fix finite $F \subset \mathcal{E}$ and a with $r = G((a, \infty)) > 0$. Resample t_F independently from $G(\cdot \mid t > a)$, keeping everything else, including V , fixed; denote the resulting field by X' . There is a joint limiting coupling with old marginal ν and new marginal $\nu' \leq r^{-|F|}\nu$. Its underlying pair marginal is exactly this original-coordinate coupling. On $\max_F t < a$, any entirely uncapped old limiting ray avoiding F is the selected new limiting ray from the same root.*

Proof. The new original marginal has density $r^{-|F|} \prod_{e \in F} \mathbf{1}_{\{t'_e > a\}} \leq r^{-|F|}$ relative to the old iid law. Recompute every finite-line graph and cocycle with the same α in the two fields. The density bound survives these pushforwards and averaging over α . Joint tightness follows from the fixed underlying pair law, compact graph coordinates, and cocycle increments bounded by cap distances; the new distance laws are dominated by a fixed multiple of the old ones. Take a joint subsequential limit. Domination passes first for nonnegative bounded continuous functions, then for all Borel sets by regularity.

Cap coordinates agree with the cap of the retained originals in the limit. Indeed a shortest circuit and all potentially shorter circuits depend on finitely many indicators. None of their originals equals M almost surely, so L_e is locally constant and s_e continuous at almost every original configuration. Absolute continuity supplies the same fact for the new marginal.

For selected-ray stability fix a root v , a finite desired prefix and $\varepsilon > 0$. By finite influence choose R containing F, v so the cap-change set lies in Q_R with probability at least $1 - \varepsilon$. Choose $S \geq R$ so all old geodesics between vertices of Q_R stay in Q_S with probability at least $1 - \varepsilon$. There

are finitely many endpoint pairs; coercivity gives such an S . Both failure bounds are uniform over the approximants because their underlying marginal is fixed. On this event a path from v cannot return to Q_R after leaving Q_S .

Realize the joint convergence by Skorokhod. Every edge of the asserted old ray satisfies the strict inequality $t_e < K_e$ almost surely; equality has probability zero by the conditional nonatomic variables. Its finite prefix through the first exit from Q_S agrees in late old graph approximants, and the strict margins and avoidance of F persist. On $\max_F t < a$ the change is upward. This prefix is unchanged; on the two localization events the rest of the finite target path is outside Q_R and unchanged as well. The whole finite path remains the unique new minimizer to the same target.

If the limiting desired prefixes differed, they would eventually differ in the discrete approximants. On the asserted limiting event this forces a localization failure in every sufficiently late approximant. Fatou bounds the bad limiting probability by 2ε . Let $\varepsilon \downarrow 0$ and intersect over roots and prefix lengths. This does not assume almost-sure eventual localization in the Skorokhod realization. $\square$

Remark 4.2. The same proof applies to a kernel changing a fixed deterministic finite set of original coordinates whose output marginal has a bounded density relative to the underlying product law, on an event where that kernel increases the originals. The joint underlying law is fixed across the approximants, the density bound gives the same domination, and strict uncapping plus the two tight localization events give the identical stability conclusion. If its upward event is only measurable, include that event's indicator in the joint lift. Its discrete convergence makes the event persist in the approximants, where monotonicity holds; the entire underlying pair-and-indicator marginal is fixed at every approximation and hence in the limit. In particular, same-status resampling has output density one, while each finite tilt has a bounded product density. This extends the explicit reference and label couplings below to those kernels; it does not assert cap conditional finite energy or stability under arbitrary uncontrolled modifications. The later edges and regions are deterministic once their site and radius are fixed; there are only countably many such choices. A random support is not being covered by an unstated uniform kernel bound.

The line convention is compatible with the circuit shortcut: if a rounded target vertex lies strictly inside a lattice circuit, its half-open unit cell is inside the polygon. The line enters the polygon and exits at a boundary edge or vertex, where rounding gives a boundary target vertex. A cheap boundary arc therefore reaches a valid target. This also justifies the uncapping argument for the precise DH approximants.

4.1 W: the cubic wall and its diagonal version

Suppose the limiting graph is not coalescing. Cut the extracted triple at last contacts with a sufficiently far common level, after all capped edges. Countably many choices of roots, level and orientation, followed by stationarity, give a deterministic positive-probability event of three wholly uncapped rays rooted on that level and strictly to its positive side afterward. Selector consistency follows from outdegree one and merging.

For an axis level take roots $(0, j_1), (0, j_2), (0, j_3)$ with $0 < j_1 < j_2 < j_3 < m$, and let F_m consist of the rungs $(-1, j)(0, j)$, $1 \leq j \leq m-1$. Raise their originals above $H_m = 4Mm^3$. A protecting open circuit enclosing any rung has crossings of $x = -1/2$ on both sides of it. The intermediate crossings are all closed, so the vertical span is at least m . Hence $L'_e \geq 2m$ and $K'_e \geq 8Mm^3 > H_m$; the new rung cap is larger than H_m .

With probability tending to one, an open circuit strictly surrounding this wall has at most $C_0 m^2$ edges. It avoids the raised set and costs at most $C_0 M m^2 < H_m$ for large m . Choose such a deterministic m with positive joint probability of the triple and cheap circuit, then choose finite

$a \geq H_m$ so that $\max_{F_m} t < a$ still leaves positive probability. Resample above a as in Lemma 4.1. The three selected rays persist.

The outer two rays and the level segment joining their roots separate the plane. A vertex outside the cheap circuit and outside the right-hand corridor cannot be in the middle class: a path to a sufficiently late middle-ray vertex would meet an outer ray, merging with a different class, or cross a rung. In the latter case its segment between two contacts with the cheap circuit can be bypassed at smaller cost. Thus the middle class lies in the corridor plus a finite set. It is bounded on the left and meets every fixed vertical column finitely, since the bounding rays eventually lie beyond every such column. It has a least vertex in lexicographic (x, y) order.

For a diagonal level use $h = x + y$, $u = x - y$, with roots $(k, -k)$ on $h = 0$. Raise both incoming edges from $h = -1$ into each $(k, -k)$, $|k| \leq m$. Their crossings of $h = -1/2$ have $u = 2k - 1/2$ and $u = 2k + 1/2$: consecutive half-integers from $-2m - 1/2$ to $2m + 1/2$. A protecting open circuit must cross on both sides of this blocked interval; its u -span exceeds $4m$, so $L'_e \geq 4m$ and $K'_e \geq 64Mm^3 > H_m$. The same $H_m = 4Mm^3$ pays the $O(Mm^2)$ outer circuit. Entry into the corridor through the interior of the level segment uses one of the raised incoming edges; entry at an outer root meets an outer ray. Lexicographic (h, u) order gives the least vertex. This parity calculation is explicit; a 45-degree rotation is not being treated as a square-lattice symmetry. Actual square rotations and reflections cover the remaining six normals.

Under the stationary old graph law no class can have a least vertex in a fixed translation-invariant order. Send one unit from each vertex to its class's least vertex if one exists. Outgoing mass is at most one, whereas that least vertex receives infinitely many units from its forward ray. The mass-transport identity and countability rule this out. Domination $\nu' \leq r^{-|F|}\nu$ transfers the null event to the new law, contradicting the construction. The limiting graph therefore coalesces. This is the replacement of DH Section 6 surgery and, later, of AH 9.10 Case 1; no cap finite energy has been invoked. The ceiling condition of DH Lemma 6.4 is vacuous here because the cap has unbounded essential support.

4.2 Deterministic reference families and the explicit coupling

For *each tangent functional* ρ , choose a point of $\partial\{\mu \leq 1\}$ at which $\{\rho = 1\}$ is the unique supporting line. This is AH's meaning of tangent; it is precisely the differentiability hypothesis of DH Corollary 4.7. That result makes the limiting cocycle slope deterministic ρ . Once the graph coalesces its calibrated cocycle is its actual Busemann function, by evaluating at a common future vertex.

We reuse the proofs of AH 5.3–5.6, not its finite-energy-dependent existence statement: conditional quantiles of the ordered ray law give, for almost every fixed quantile, a measurable translation-covariant coalescing family of slope ρ . The already proved graph coalescence supplies the required premise. The proof of AH 4.4 identifies any two such families. Its boundary expectation step (AH 4.6) applies to the cap because coordinate, or diagonal zigzag, boundary routes have finite mean and boundary translations are ergodic. S's boundary-comparison appendix is unnecessary.

Fix one good quantile. Uniqueness for every other good fixed quantile, followed by Fubini, says that the conditional ray law given s is a point mass. Countability of roots identifies the entire graph as a measurable family $G_\rho(s)$. This statement is conditional on s , not merely on the richer X . Domination transfers it to the modified graph. Apply Lemma 4.1 at late vertices beyond capped edges and F . The old and new G_ρ rays share tails. Since both graphs are now functions of their cap fields and the projected pair marginal is the prescribed law of (X, X') , this stability holds under that *explicit original coupling*, not only along a chosen subsequence. Four square rotations provide the reference families needed for the AH order construction.

5 Ordered labels and stability of selected rays

All uses of AH Sections 6–10 in this section are *proof-level reuse*. AH's standing A1/A2 assumptions include finite-energy conditions not satisfied by s . Section 8 lists each use and its replacement premise; we do not apply its final structural theorem as a black box.

The reference families, uniqueness and planar order give AH Section 6.1's cyclic ray order. Sections 6.2–6.4 average source flows over independent auxiliary Voronoi cells. The product structure is the cap law times those auxiliary uniforms, not independence of capped edges. The conditional-mean construction in Sections 6.5–6.7 gives monotone labels consistent under coalescence. Section 7 uses planar trapping, ergodic density and extended shape to control noncrossing selectors. These arguments introduce no edge resampling. The one downward operation occurs in AH 8.6.

5.1 The replacement for AH 8.6

For an edge e , independently resample t_e conditional on its open/closed status, leaving all V fixed. This kernel is reversible for the underlying product law. Every status, hence every L_f and K_f , stays fixed; only s_e can change. If e is usable and uncapped, its cap decreases by more than $\delta = g/2$ with probability at least p_* , by sampling its low band. The fixed δ, p_* are chosen before the root and finite witness paths in AH's bad event.

Let $\mathcal{F}_n$ be the originals and auxiliaries on edges with both endpoints in Q_n . For annular edges outside this set, the kernel preserves $\mathcal{F}_n$ and is reversible conditionally on it. Let $A = A(v, P_1, P_2, P_3, \delta)$ be AH's finite-witness event, with its witness paths inside Q_n . Its deterministic Lemmas 8.4–8.5 use exactly one decreased edge: a successful decrement on the relevant extremal ray g_1 destroys A and places that edge on the new clockwise extremal. They therefore apply to this exact single-cap-edge change.

Let $C_1(n)$ be the uniform length event of Lemma 3.1, and let $C'_1(n)$ use $3Ln$ in place of Ln . Lowering one edge implies $C_1 \Rightarrow C'_1$: a new geodesic using the edge splits into two old-minimal pieces with endpoints in Q_{2n} , so has at most $2Ln + 1$ edges. An avoiding path is already old-minimal. Usable-edge counting in the original iid field gives at least half usable edges on every long simple path from the box, with probability tending to one. All v -rays are uncapped beyond finite E_v . Their crossings of the annulus thus have at least γn eligible edges, say $\gamma = 1/3$, on an event $C_2(n, v)$ of probability tending to one.

Writing D_e for eligibility, conditional reversibility gives

$$p_* \mathbb{P}(A \cap C_1 \cap D_e \cap \{e \in g_1\} \mid \mathcal{F}_n) \leq \mathbb{P}(A^c \cap C'_1 \cap \{e \in g_1\} \mid \mathcal{F}_n).$$

Sum over annular edges. On C'_1 the part of g_1 between its first and last visits to the outer box has at most $3Ln$ edges. Consequently

$$\mathbb{P}(A \mid \mathcal{F}_n) \leq \mathbb{P}(C_1^c \mid \mathcal{F}_n) + \mathbb{P}(C_2^c \mid \mathcal{F}_n) + \frac{3L}{p_* \gamma} \mathbb{P}(A^c \mid \mathcal{F}_n). \quad (4)$$

The two error terms tend to zero in probability, whereas martingale convergence makes the conditional probability of A tend to $\mathbf{1}_A$. If $\mathbb{P}(A) > 0$, (4) is impossible on a positive-probability part of A for large n . Thus $\mathbb{P}(A) = 0$. Countability of the finite witnesses yields AH 8.2 with its original simultaneous quantifiers.

The remaining proofs of Sections 8.2–8.4 now use order, compact limits and countability to give a deterministic closed label set $\mathcal{L}_*$, the two extremal selectors at each fixed label, and their continuity properties. The proof of AH 9.2 constructs a stationary increment/path extension. Lemma 3.2, rather than an assertion of extension ergodicity, provides its linearity. Its supporting-arc argument and deterministic selector direction set yield one of the eight eventual half-planes.

5.2 S: common prediction sites and symmetric label transfer

Consider the original upward coupling of Lemma 4.1. It is generally not stationary as a joint law, but both marginals are absolutely continuous with respect to the original law. Its cap difference set is finite. Reference stability from Section 4, together with the geometric proof of AH 9.7 and extended metric/Busemann bounds, gives a translate of a cone of positive deterministic angle in which the two bounding reference rays agree and the sector contains no changed edge. For agreement, the old and new reference rays avoid the change set and share a tail; their prefixes to a common tail vertex have the same costs in both fields, so finite uniqueness identifies them. This replaces the specific AH 9.6 conclusion used in the proof of 9.7.

Fix a label and the finite neighboring-selector list used in AH 9.8. Choose a finite stub length so neighboring unequal labels have different stubs with probability at least $1 - \eta$, then a finite cap-patch radius whose maximum-likelihood predictions of those stubs are simultaneously correct with probability at least $1 - \eta$, adjusting η for the finite list. Martingale convergence supplies this prediction property; fixed tie-breaking makes the predictors translation-covariant. Ergodicity gives bad-site density at most η in each marginal on a full event. Absolute continuity transfers that full event to both fields. Their joint bad-site union therefore has upper density at most 2η , even though the coupling is not stationary. Choose 2η below the cone's density, and discard the finitely many patches meeting the changed set. Some cone site is good in both fields and has identical patches, hence identical actual selector stubs.

The planar comparison in the proof of AH 9.8 now traps new extremal labels between their prescribed old neighbors. New realized labels belong to $\mathcal{L}_*$ by absolute continuity of the new marginal. If α is approached from below, use a countable sequence of lower labels. If it has an immediate predecessor β , the strict comparison “new label $> \beta$ ” and membership in $\mathcal{L}_*$ imply new label $\geq \alpha$. Use the symmetric successor convention above; reference-equality cases are handled directly. This avoids a nonexistent approximating sequence at an isolated label. Monotone squeezing transfers every intervening α -ray, not only the two extremals.

The premises of this label comparison are symmetric: both marginals are absolutely continuous, the cap difference is finite, and corresponding reference rays coalesce. Apply it in both orientations. The reverse-label statement does not require a reverse upward resampling kernel.

Lemma 5.1 (Finite-change tails). *If two positive coercive metrics w, w' agree off finite F_0 , every w -ray has a w' -geodesic tail, and conversely. The cutoff is uniform over rays from a fixed root.*

Proof. If F_0 is empty the assertion is immediate. Otherwise let V_0 contain the endpoints of F_0 , let $\Delta = \sum_{F_0} |w'_e - w_e|$, and put $C = \max_{v \in V_0} T_w(x, v)$. A simple competitor from a late a to a later b on an x -ray that meets V_0 has new cost at least

$$T_w(x, a) + T_w(x, b) - 2C - \Delta = T_w(a, b) + 2T_w(x, a) - 2C - \Delta.$$

Use the first and last contacts and triangle inequalities for this bound. Once $T_w(x, a) > C + \Delta/2$ and the tail avoids V_0 , it cannot improve the unchanged ray segment. Avoiding competitors have identical costs. Positive weights permit loop erasure before the one-copy Δ estimate. Coercivity supplies the uniform late cutoff; reverse the metrics for the converse. $\square$

5.3 Entire extremal selectors and the second wall application

Suppose an old extremal α -ray from v is wholly unchanged under $s' \geq s$. It is a new ray by monotonicity. If the new extremal differed on the relevant side, Lemma 5.1 and the reverse label comparison would make its tail an old α -ray. The proof of AH 9.9 then says old finite geodesics

from v to late vertices of that tail converge to the old extremal. Put v and the cap changes in Q_R , and take Q_S containing all old geodesics between vertices of Q_R . Once such a finite geodesic agrees with the old extremal through its first exit from Q_S , it is wholly unchanged: a later return to Q_R is impossible. Finite uniqueness makes it strictly cheaper than the different new-extremal prefix in s , and its unchanged cost is still strictly cheaper in s' , a contradiction. Thus the entire selected ray is preserved. Prefix convergence alone would not have sufficed without this no-return argument.

Apply the wall of Section 4.1 to late uncapped extremal triples. The selector premise is now proved by the preceding paragraph, rather than left as a historical condition. AH 9.11's mass transport gives coalescence. The remaining proofs of 9.1 and 9.12–9.13 use deterministic labels, stationarity and disintegration, without a new energy operation.

5.4 The all-ray output needed by S

The proof-level chain in AH Section 10, detailed in Section 8, now supplies a deterministic closed set $\mathcal{C}$ of linear functionals dominated by μ . On one event every cap ray p has a Busemann function B_p with

$$B_p(0, x) = \rho_p(x) + o(|x|), \quad \rho_p \in \mathcal{C},$$

uniformly at infinity; its limiting directions lie in the contact arc $\{u : \rho_p(u) = \mu(u)\}$. The uniform all-ray cone bounds come from countably many finite reference nets, not an uncountable intersection of fixed-ray assertions. For each *fixed* functional there is a unique coalescing, backwards-finite family; its null set may depend on that functional. Backwards finiteness uses the mass-transport proof of AH 4.5. The presence of every tangent functional uses the explicitly quantified construction in Section 4. All these statements concern s and its Busemann functions, not uniform asymptotics for the original t .

6 P: independent splices and the changed-vertex barrier

S Section 3 uses only square-lattice rotations and reflections for its charts. Fix one chart with upper labels having contact directions strictly above the horizontal axis and lower labels strictly below it. These are D4 isometries, so coordinate Lipschitz constants and nearest-neighbor height counts are unchanged. The diagonal wall coordinates used earlier are a separate construction. The angular gap may depend on a label; no uniform gap over the label set is asserted.

S Section 4 transfers exceptional labels by independent half-plane pasting. Its exact shared-weight premise fails for caps. We now prove its required transfer conclusion with actual changed vertices in place of the cut. Let E_n^+ contain edges whose two endpoint heights are at least $-n$. Form X_n^+ from X on E_n^+ and an independent field elsewhere; form X_n^- from another independent field on E_n^+ and X elsewhere. Splice both originals and cap auxiliaries. For each fixed n , the two outputs are independent and each has the original product law. Their caps s_n^+, s_n^- are therefore independent with the cap law. Independence across n is not needed.

Let D_n^+ be all endpoints of edges where s and s_n^+ differ, and define D_n^- similarly. If an unchanged original edge has protecting minimum circuits in both fields entirely inside the shared region, the two minima are mutually admissible and its caps agree. Thus, with $|e|$ the midpoint norm, every changed vertex satisfies

$$\begin{aligned} z \in D_n^+ &\implies z_2 \leq -n + 3 + \max_{e \ni z} (L_e(X) + L_e(X_n^+)), \\ z \in D_n^- &\implies z_2 \geq -n - 3 - \max_{e \ni z} (L_e(X) + L_e(X_n^-)). \end{aligned} \tag{5}$$

The constants cover endpoint and midpoint conventions. These sets can be infinite and need not be curves or connected.

Define

$$A_n(X) = \sup_e \frac{L_e(X)}{|e| + n}, \quad E_n(T) = \sup_{x,y} \frac{|T(x,y) - \mu(y-x)|}{|x| + |y| + n}.$$

Uniform sublinear coding radii and (3) imply that both tend to zero almost surely. The same quantities for X_n^+ tend to zero in probability, since each has the same marginal law. Choose one deterministic subsequence n_k on which their probabilities of exceeding 2^{-k} are summable. Borel–Cantelli gives, on that subsequence, all four errors tending to zero. Consequently for random $\varepsilon_k \downarrow 0$,

$$z \in D_{n_k}^+ \implies z_2 \leq -n_k + \varepsilon_k(|z| + n_k). \quad (6)$$

For every fixed n , both fields' coding radii also give $z_2 \leq o(|z|) + O_n(1)$ on D_n^+ and the reflected bound on D_n^- , simultaneously for the countable collection of n .

Lemma 6.1 (Moving and fixed barriers). *Every finite collection of upper-ray Busemann witnesses in s transfers to $s_{n_k}^+$ for all sufficiently large k , with the same exact labels. For every fixed n , every lower ray has a tail in s_n^- with its exact original label. These statements hold pathwise for all rays on the structural full event.*

Proof. Fix an upper ray p , its functional ρ , and a finite set of basepoints K containing p_0 . Compactness and the strict upper contact sector give $\eta, \gamma > 0$ such that

$$\mu(d) - \rho(d) \geq \gamma|d| \quad \text{if } d_2 \leq \eta|d|. \quad (7)$$

For large k , (6) implies $|z| \geq n_k/2$ for all $z \in D_{n_k}^+$. Uniformly for $x \in K$, $|z - x| \geq |z|/2$ and $|x| + |z| + n_k \leq 8|z - x|$. Also $(z - x)_2 \leq 3\varepsilon_k|z| + O_K(1)$, so (7) applies. The two metric errors and the Busemann error are small compared with $|z - x|$. Hence, uniformly on the infinite changed set,

$$\min(T, T_n^+)(x, z) - B_p(x, z) \geq (\gamma/2)|z - x| \geq \gamma n_k/8. \quad (8)$$

The ray itself avoids $D_{n_k}^+$ for large k : its eventual strict upper sector excludes distant vertices by (6), and the receding cut excludes its finite prefix.

For a walk from x to p_j meeting $D_{n_k}^+$, let z be its *last* vertex in that set. Its suffix has exactly the same edge costs in both fields. Calibration and the metric bound for B_p show that the new walk costs at least $T_n^+(x, z) + B_p(z, p_j)$, which exceeds $B_p(x, p_j)$ by the barrier. The same bound holds in the old field. Walks avoiding the changed set have exactly equal costs. With $x = p_0$ this preserves all prefixes of p . For any fixed $x \in K$ the calibration defect $T(x, p_j) - B_p(x, p_j)$ tends to zero, so eventually both minimizers avoid the changed set and

$$T_n^+(x, p_j) = T(x, p_j). \quad (9)$$

This preserves the requested finite Busemann witnesses. The subsequence was chosen before p , so the argument is simultaneous over rays without new probability intersections.

To identify the exact label fix n and a preserved upper ray. For any $c > 0$, take $x_2 \geq c|x|$. The fixed-cut bound gives $z_2 \leq \delta|z| + C_\delta$ on D_n^+ . For small δ and large $|x|$, elementary triangle inequalities yield

$$|z - x| \geq \kappa(|x| + |z|), \quad \kappa = c/(c + 8).$$

Indeed $|z - x| < c|x|/4$ would imply $|z| < 2|x|$ and $x_2 - z_2 > c|x|/2$, a contradiction when $\delta \leq c/8$. Decrease δ so $\delta/\kappa < \eta/2$. The displacement lies in the cone of (7); the fixed-metric and Busemann errors then give

$$\min(T, T_n^+)(x, z) - B_p(x, z) \geq (\gamma\kappa/4)(|x| + |z|)$$

for all sufficiently far upper-sector x , uniformly over $z \in D_n^+$. The last-changed-vertex argument gives equality of Busemann values at those x . Use $x = ke_2$ and $x = k(e_1 + e_2)$ to identify the linear functional in the new field as exactly ρ . The reflected argument applies for each fixed n to a sufficiently late tail of every lower ray in s_n^- , using two independent lower directions. Thresholds may depend on the ray and on n , as in S's transfer statement. $\square$

S Section 4's remaining proof uses the countability of exceptional labels, Borel/analytic witness sets and independence of the two output fields. Lemma 6.1 supplies exactly its two transfer quantifiers, including finite witnesses and exact labels; fixed-label coalescence is available from Section 5.4. Its four-witness independence argument therefore gives opposite exceptional coalescence simultaneously. No exact half-plane agreement of capped weights and no original Busemann linearity have been assumed.

7 Status-preserving budgets and the final contradiction

We spell out the probability-kernel replacements and the precise deterministic inputs reused from S Sections 3–8. Their charts, label budget, finite-tree count, posterior selection argument, shared-union detour and final filters are antecedents, not new claims. All events below live on (t, V) and use cap distances and cap labels. Remark 4.2 supplies reference and label stability for the finite kernels used here on their upward events.

7.1 Charts, comparison, and the one-edge budget

S Section 3's deterministic chart construction gives countably many compact label charts I , each with a horizontal order parameter $r(\rho)$, finite width $W_I = r(\rho_r) - r(\rho_\ell)$, and slope $\sigma > 0$. The arm cones are

$$Q^\pm(k, H) = \{z : \pm(z_2 - k_2) \geq \sigma|z_1 - k_1| - H\}.$$

Choose nested finite deterministic endpoint sets D_j dense in chart order and containing the endpoints of every complementary gap. Consecutive endpoints define the bins Π_j . Their widths sum to W_I , and distinct nonendpoint labels eventually separate. Fixed-label backwards finiteness excludes bigeodesic end labels at the countable union of those endpoints.

Here is the deterministic comparison from S Section 5. Suppose a finite increase near a site k adds total cost at least b on finite initial portions of a bigeodesic whose upper label lies in a bin (α, β) . Suppose its arms have cone offset H_p , and the base upper α and lower $-\beta$ comparator rays from y have offset H_c . If the change and gain portions lie in the radius- R box about k , then for

$$y_1 - k_1 > S = \max \{R + (R + H_c)/\sigma, (H_p + H_c)/\sigma\}$$

the comparator rays avoid the changed box, remain geodesics with their labels, and the chart crossing argument gives

$$b \leq B_\alpha(y, k) + B_{-\beta}(y, k). \quad (10)$$

This uses finite uniqueness, cone crossing and calibration. Successive comparisons telescope, leaving only the horizontal label width. The cap has every one of these premises by Sections 5–6.

At a grid edge e , independently replace the original t_e by a sample from its status-conditional law. Each individual output has exactly the original iid law, with the same independent V . All statuses and thresholds stay fixed, so only that cap edge changes. Conditional on the *entire* individual output and V , its hidden old value is in the low band of its status i with probability $p_{a,i}/r_i \geq p_*$. On a usable output bigeodesic the edge is wholly uncapped by Proposition 2.2. Mark when its

hidden old value is low. The old value is below $a_i < b_i$ and the common threshold exceeds the output value, so the cap increase is at least g and no other cap changes.

The source's base-comparator regularity check uses both endpoints of e : a base ray either avoids it or uses it first, in which case its tail starts at the other endpoint. Monotonicity and label transfer give the required base cone offset from output regularity. Thus its spacing $\ell(H) \leq C_0(H+1)$ and (10) apply. If $A_e^{\alpha,\beta}(H)$ denotes a usable, H -regular output bigeodesic event with label in that bin, the mark probability and budget imply

$$\mathbb{P}(A_e^{\alpha,\beta}(H)) \leq \frac{\ell(H) [r(\beta) - r(\alpha)]}{gp_*}, \quad \mathbb{E}N_e(H) \leq C(H+1). \quad (11)$$

The latter follows by partition refinement and monotone convergence; $N_e(H)$ counts distinct usable regular edge labels. Independence of marks is not needed.

S Section 6's rooted finite-tree leaf count credits at least $nF_n/4$ distinct usable edge-label pairs to F_n forks. Its usable-path estimate is valid here in the *original* iid field: a path of length j has at least half bad edges with probability at most $2^j \epsilon^{j/2}$, and the path count contributes at most 4^j , giving $(8\sqrt{\epsilon})^j$. Cap bigeodesics are wholly unchanged. The simultaneous cap cone bound and (11) therefore provide exactly its expectation bound. First let $n \rightarrow \infty$ at fixed cone error and then let that error vanish. There are no forks. With opposite exceptional coalescence from Section 6, the source's deterministic conclusion identifies the same two Busemann functions for all bigeodesics of a common random label, simultaneously. This is not an intersection of fixed-label null events.

7.2 Finite raises, likelihoods, and posterior independence

Assume bigeodesics have positive probability. The source's countable chart and cone-offset reduction gives fixed I, σ, H_0 and $p > 0$ for a bigeodesic through the origin with arms in $Q^\pm(0, H_0)$ and nonendpoint label in I . Shrink this event, if necessary, by fixing a usable-prefix threshold n_0 ; the path count above makes the discarded probability arbitrarily small. Retain the notation p for the resulting positive probability. These choices are made before the requested toll m .

Let $h(e) = \max(|w_2|, |w'_2|)$ for the endpoints of e , and put

$$\delta_h = \frac{\eta}{(h+2)\log(h+2)}, \quad A_k(R) = \left\{ e : h(e) \leq R, \max_{w \in e} |w_1 - k_1| \leq 1 + \frac{h(e) + H_0}{\sigma} \right\}.$$

Take independent trigger uniforms and samples $Z_{e,i} \sim G(\cdot | H_i)$, independent of X and across edges. If $t_e \in L_i$, raise it to $Z_{e,i}$ with probability $\delta_h/p_{a,i}$; otherwise keep it. Choose η so $\delta_0 \leq \min_i p_{a,i}/2$. Use the same sample for every region containing e , applying it only once in overlaps. All statuses and all V remain fixed, so all K_e remain fixed. Cap changes are confined exactly to the raised edges of the specified finite region.

The one-edge output density relative to G is

$$\lambda_h(t) = 1 + \sum_{i=o,c} \left[-\frac{\delta_h}{p_{a,i}} \mathbf{1}_{L_i}(t) + \frac{\delta_h}{p_{b,i}} \mathbf{1}_{H_i}(t) \right], \quad (12)$$

$$\int \lambda_h dG = 1, \quad \int \lambda_h^2 dG = 1 + \delta_h^2 \sum_{i=o,c} (p_{a,i}^{-1} + p_{b,i}^{-1}).$$

There is mass δ_h moved in each status, not just in one status. The number of region edges at height h is at most $(12 + 8/\sigma)(h + H_0 + 1)$. Independence gives a product likelihood L with

$$\mathbb{E}L^2 \leq \exp(C\eta^2), \quad \text{TV}(\mathcal{L}(Y^{k,R}), \mathcal{L}(X)) \leq \frac{1}{2} \sqrt{e^{C\eta^2} - 1},$$

uniformly in k, R , because $\sum_h (h + H_0 + 1) / ((h + 2)^2 \log^2(h + 2)) < \infty$. Choose η so this is at most $p/8$. Finite union fields are also absolutely continuous. Cap pushforward cannot increase total variation.

Conditional on the whole original output Y and all cap auxiliaries V , the hidden success indicators are independent, with

$$\theta_e = \sum_{i=o,c} \mathbf{1}_{H_i}(Y_e) \frac{\delta_{h(e)}}{p_{b,i} + \delta_{h(e)}}. \quad (13)$$

Indeed the joint success/output measure in H_i is $(\delta_h/p_{b,i})G(dt)$; divide by (12). The product conditional law extends from finite output cylinders to the full output sigma-field. Circuit lengths are already functions of the output statuses, so they introduce no additional conditioning. A witnessing cap bigeodesic is selected from (Y, V) alone. On a successful usable edge of it, $Y_e = s_e \leq K_e$ and the old original is at most $a_i < b_i \leq Y_e$; the old cap is the old original too. Thus the cap gain is at least g .

For every prefix length at least n_0 , at least half its edges are usable. The j th upper-arm edge has $h(e) \leq j$, so Abel summation and (13) give a conditional expected number of successes bounded below by a positive constant times $\sum_{j=n_0}^R ((j+2) \log(j+2))^{-1}$, which diverges. Conditional variance is at most the mean. Fix a near radius R_0 with mean at least eight. For any toll m , choose $R(m)$ with mean at least $\max(2\lceil m/g \rceil, 32)$. Chebyshev bounds give, jointly, a near strict raise and total gain at least m with conditional probability at least $3/4$. The source's marking calculation then gives probability at least $p/4$. The horizontal spacing

$$\ell > R_0 + 1 + (R_0 + H_0)/\sigma$$

is fixed independently of m . Posterior independence is not asserted after the later base or neighbor filters.

7.3 Shared unions and telescoping

A bigeodesic in the individual region at k has every edge of height $\leq R$ inside $A_k(R)$ by its cone width; higher edges are never raised. Passing to the two-site union therefore leaves all of its edges unchanged, while competitors only increase. The same holds for the selected path at the second site. A near raised edge of the second path lies outside the first region by the fixed spacing, so in the first field it still has its base cap cost and in the union its cost is strictly larger. The common Busemann pair for equal labels and the source's vanishing-excess detour through this strict edge rule out repeated marked labels. This is exactly the geometric use of the near raise; the large radius pays only the toll.

S Section 8's base-comparator filter loses at most $p/8$, leaving $q_m \geq p/8$. For each fixed m , the comparison separation $S(m)$ is finite. Refine Π_j and discard marks with a same-bin neighbor within $S(m)$. Distinct labels and finitely many neighbors make this filter disappear pointwise as $j \rightarrow \infty$. Apply (10) successively in each bin; Busemann increments telescope and their linear parts leave the sum W_I of bin widths. The bound is $q_{m,j} \leq \ell W_I/m$. Bounded convergence therefore yields

$$p/8 \leq q_m \leq \ell W_I/m \quad \text{for every integer } m.$$

The constants p, ℓ, W_I were fixed before m , a contradiction for large m . Sites, finite regions, unions, partition endpoints and chosen radii form countable collections, so all inherited full events can be imposed simultaneously. Random-label equalities were already pathwise. There are no cap bigeodesics. Proposition 2.2 proves Theorem 1.1 for the remaining laws, and S covers (1).

8 Proof-use crosswalk and verification limits

The following map records proof reuse, not the invocation of a theorem whose standing finite-energy hypothesis is absent. In each row the original geometric or order argument is retained only after the indicated replacement has been supplied. In particular the cap law is never asserted to satisfy AH A2(v) or A2(vii). All probabilities for coordinate surgery are taken on the underlying product space, including the independent cap marks.

Original use	Hypotheses and replacement in this deduction
DH Sections 3–5	A2' supplies the averaged rounded-line graphs, cocycles, outdegree, direction and supporting-arc conclusions. Section 3 proves the required cap shape and integrability. The lifted extension is not assumed ergodic; Lemma 3.2 handles its invariant random slope.
DH Lemma 6.2 and triple extraction	These occur under the Section 6 standing upward-finite-energy assumption. Only their planar extraction proofs are reused: stationarity, directedness and positive ergodic density give three disjoint rays in one of the eight eventual half-planes. The probability-changing step is Lemma 4.1 and Section 4.1, not the DH finite-energy conclusion. DH 6.4's bounded-ceiling issue is vacuous for this unbounded cap.
AH 4.4, 4.6; 5.3–5.6	Fixed-slope uniqueness uses cap boundary integrability. Conditional quantile extraction uses the coalescent graph just constructed; it does not supply coalescence. Point-mass conditional graph laws descend the joint limiting coupling to the explicit underlying-coordinate pair.
AH 6.1–6.7	The auxiliary partitions and labels use independent auxiliary randomness, stationarity, shape, finite uniqueness and the four reference families. Independence of cap edge weights is not used. References are constructed in Section 4.
AH Section 7	The noncrossing order, extremal rays and limiting labels use the preceding references, planarity, uniqueness, measurability and shape. They are reused at proof level in Section 5.
AH 8.3–8.6	The single-edge downward step is replaced by same-status resampling. It changes exactly one cap coordinate, has a reversible underlying law, and lowers an uncapped good edge by a fixed positive amount with uniformly positive probability. The annular good-edge density, eventual uncapping, conditional estimate and martingale contradiction are given in Section 5, including equation (4).
AH 8.1–8.4, remaining conclusions	The resulting deterministic closed label set and extremal selectors follow by the original order/countability proofs after the previous row. Membership of new labels in that same set is transferred by absolute continuity. Isolated predecessors and successors are treated explicitly, not by nonexistent monotone approximating sequences.
AH 9.2–9.4	Random-extension cocycle linearity follows from Lemma 3.2; directional ergodicity is not inferred from cap mixing. The order and supporting-cone consequences then use that linearity and shape.

Original use	Hypotheses and replacement in this deduction
AH 9.5–9.9	Common local reference stubs have positive density under the two absolutely continuous marginals. Comparison is symmetric; squeezing extends it to every ray with the label. Lemma 5.1 supplies the finite-change tail map. The final no-return argument upgrades preserved tails to the selected whole ray on the stated upward, unchanged-path event.
AH 9.10, 9.11	The wall operation is Section 4.1, with cubic cap, cheap outer circuit, exact diagonal crossings and least-class mass transport. The required selector stability is established before this reuse. The subsequent coalescence deduction uses the original planar argument.
AH 9.1, 9.12–9.13; Section 10	These are reused only after the structural and modification premises above. Countable label approximations, planarity, coalescence, shape and fixed-functional uniqueness give the all-ray output stated in Section 5.4; fixed-label null events are not silently intersected over an uncountable random set.
S Section 04	Product independence belongs to spliced originals and marks. Section 6 proves moving-witness and fixed-cut transfers across the changed-cap band, including equality of the late suffix costs and labels. The remaining exceptional-label countability argument is the source’s.
S Sections 05–08	Section 7 keeps the budget, fork, grid, harmonic tilt, neighbor filter and telescoping operations. Two-status kernels replace iid cap resampling. Their exact densities and posterior probabilities are equations (12)–(13); statuses and cap radii remain fixed. No new contribution is claimed for the source’s counting argument.

Remark 4.2 explicitly generalizes the fixed-threshold coupling to fixed finite-support kernels with bounded output density. The underlying pair (and, when needed, its upward-event indicator) is retained in the lift; its projected law is identical at every approximation and in the limit. Selected-path stability is restricted to the upward event. Each later support is deterministic after the specified integer site and radius are fixed. Countably many choices may then be intersected; arbitrary state-dependent supports or unbounded-density kernels are not covered by this statement.

8.1 Limited deterministic formal support

The accompanying frozen Lean support has 14 support declarations and 17 control declarations. Its principal all-walk statement concerns a graph with a nonnegative changed weight function, a finite set of changed edges, a finite absolute-change budget, calibrated old ray segments, radial additivity and escape, and a bound on old root distances to changed-edge endpoints. It proves that sufficiently late segments minimize over all finite walks. The passage from edge-simple competitors to walks uses loop erasure and nonnegativity. This supports the deterministic argument of Lemma 5.1.

The formal objects do not construct an infinite random geodesic or prove the probabilistic assumptions supplying calibration, coercivity or finite influence. They do not formalize the circuit cap, shape, conditional kernels, Busemann limits, labels, coalescence, splicing, or Theorem 1.1. The finite controls illustrate hypotheses and counterexamples; their number is not a count of independently proved probabilistic lemmas.

8.2 Source-reading boundary

The author derivation uses the operative source and primary interfaces listed in the accompanying reading record. Prior proof scrutiny was bounded: in particular some AH Sections 4, 6, 7 and 10 and DH Sections 3–5 were checked at statement/interface level rather than through every proof line. This manuscript makes the required proof-level uses explicit for further scrutiny; it does not turn such bounded reading into a claim of exhaustive independent verification. Cox–Durrett, Kesten and Boivin are credited through the actually read discussions in AH, DH and Cerf–Théret. No original-body reading of those works is asserted. The public evidence records distinguish author assistance, independent reports, formal support and compilation checks.

References

- [1] OpenAI, *No bigeodesics in planar first-passage percolation*, stipulated source 212, repository revision [fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb](#). Exact source paths and blob identities are recorded in the accompanying source receipts.
- [2] D. Ahlberg and C. Hoffman, *Random coalescing geodesics in first-passage percolation*, [arXiv:1609.02447v2](#).
- [3] M. Damron and J. Hanson, *Busemann functions and infinite geodesics in two-dimensional first-passage percolation*, [arXiv:1209.3036v1](#); Communications in Mathematical Physics 325 (2014), 917–963, doi:10.1007/s00220-013-1875-y.
- [4] M. Damron and J. Hanson, *Bigeodesics in first-passage percolation*, [arXiv:1512.00804v2](#), Conjecture 1.
- [5] A. Auffinger, M. Damron and J. Hanson, *Limiting geodesics for first-passage percolation on subsets of $\mathbb{Z}^2$* , [arXiv:1302.5413v2](#); Annals of Applied Probability 25 (2015), 373–405, doi:10.1214/13-AAP999. Appendix C, especially Lemma C.5 and (26).
- [6] R. Cerf and M. Théret, *Weak shape theorem in first passage percolation with infinite passage times*, [arXiv:1404.4539v2](#).