

Spectral chart selection and average trace-class distortion

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Abstract

We construct a sequence of genuine finite negative-type metrics with n points whose first uniform-average distortion into the trace class $S_1(H)$ is at least $c\sqrt{\log n}/(\log \log n)^{9/2}$, uniformly over all complex Hilbert spaces H . The construction uses the multichart Hilbert kernel of the pinned uniform-sparsest-cut source. Its scalar coordinate selection is replaced by a dimension-independent matrix selection: universal scalar moment matching gives spectral cutoffs for all matrix tuples, and a positive Gaussian dual pairing averages these cutoffs without requiring them to commute. We track the enlarged chart number through the kernel, rounding mesh and triangle repair. Finite-rank compression, Hermitian dilation and uniform copies preserve the full statement. The result is new to this project; a bounded search found no antecedent, but literature priority is not claimed.

1 Statement, conventions and source dependence

For a complex Hilbert space H , write $S_1(H)$ for its trace-class operators, with norm $\|A\|_1 = \text{Tr } |A|$. A finite metric d has *negative type* if there are real Hilbert vectors z_x with $d(x, y) = |z_x - z_y|^2$. We require $d(x, y) > 0$ for $x \neq y$ and every triangle inequality; a squared Hilbert distance without these properties is not called a metric here. All pair averages are over two independent uniform points, or equivalently over all ordered pairs including the diagonal.

Theorem 1.1. *There are an absolute constant $c > 0$ and genuine finite negative-type metrics (V_j, d_j) , with $n_j = |V_j| \rightarrow \infty$, such that for every complex Hilbert space H and every map $F : V_j \rightarrow S_1(H)$ satisfying $\|F(x) - F(y)\|_1 \leq d_j(x, y)$ for all x, y , one has*

$$\frac{1}{n_j^2} \sum_{x, y \in V_j} \|F(x) - F(y)\|_1 \leq c^{-1} \frac{(\log \log n_j)^{9/2}}{\sqrt{\log n_j}} \frac{1}{n_j^2} \sum_{x, y \in V_j} d_j(x, y). \quad (1.1)$$

There is no dimension, separability or rank restriction on H or F .

This also gives the displayed lower bound on bi-Lipschitz distortion within negative-type metrics. For general finite metrics it is weaker than the classical cotype grid bound recalled in Section 8; no new general-metric bi-Lipschitz bound is claimed. It is an existence result, with no efficient or rational construction claim. We write C for absolute constants, including c^{-1} after adjusting c , and all logarithms are natural.

Our geometric input is OpenAI's *Near-square-root logarithmic integrality gaps for uniform sparsest cut* [1], at the immutable revision specified in Appendix A. Its theorems are taken as inputs; this paper does not audit their truth. We use its explicit common kernel, chart extension, angle repair,

interface coarea and copy mechanism. The original scalar contraction cannot simply be asserted for trace norms: its good charts are selected separately for scalar coordinates. Section 4 supplies the replacement. The other operator and probability tools are standard, and are proved here in the particular forms needed.

The chart number and exponent were chosen by the method before the theorem scope was frozen; the freeze preceded the complete proof, not the method. Specifically, set, for a sufficiently large integer m ,

$$\begin{aligned} N = m^6, \quad q = \lceil \log N \rceil, \quad S = m^{10q}, \quad a = m^{-40}, \quad b = m^{40}, \\ \varepsilon = m^{-200}/S, \quad r = m^{-8}, \quad \tau = m^{-2000}/S^2, \quad \lambda = m^{-2}. \end{aligned} \quad (1.2)$$

The denominator exponent $9/2$ is method-shaped, not an asserted optimum. Restricting (1.1) to the diagonal copy of ℓ_1 in S_1 gives a weaker commutative bound than the source's exponent 3; our theorem does not supersede that result. Section 8 treats the closest alternative operations, including the sharp metric-cotype grid bound and the pair-average inequality implicit in the cube-quotient Talagrand argument. We make no claim about the trace-class embedding of Heisenberg groups or the general quantum-expansion metric question.

2 A variance-normalized operator moment bound

All matrices in Sections 2–4 are finite complex matrices. The notation $A \leq B$ means Loewner order.

Lemma 2.1. *If $X_1, \dots, X_m$ are selfadjoint and $\sum_j X_j^2 \leq I$, then for independent standard real Gaussian variables g_j and every integer $q \geq 1$,*

$$0 \leq \mathbb{E} \left(\sum_j g_j X_j \right)^{2q} \leq (2q - 1)!! I. \quad (2.1)$$

Proof. Isserlis' identity, including repeated indices, expresses the expectation exactly as a sum over all pairings π of the $2q$ positions. For a fixed pairing, let W_π be the sum of words obtained by assigning a free index in $[m]$ to each pair and putting its X at both positions.

On a Hilbert space K carrying the matrices, define $V : K \rightarrow \mathbb{C}^m \otimes K$ by $Vv = \sum_i e_i \otimes X_i v$. Since $V^*V = \sum_i X_i^2 \leq I$, both V and V^* are contractions. Process the positions of the word from right to left. At the first encounter with a pair, apply V to the K factor and keep the new $\mathbb{C}^m$ factor as its index register. At its second encounter, permute that register next to K and apply V^* . Tensor each operation with the identity on all other open registers. Permutations are unitary, so every operation is contractive. This applies to crossing pairings as well as nested ones. At the end no registers remain.

Expanding the composition shows that opening a register sums its index, closing it enforces equality with the matched occurrence, and the order of the X factors is the original word order. Thus the resulting map is W_π and $\|W_\pi\|_{\text{op}} \leq 1$. No positivity of an individual W_π is used. There are $(2q - 1)!!$ pairings. Their sum has norm at most that number, and the sum is positive because it is the expectation of an even power of a selfadjoint matrix. This proves (2.1). $\square$

Operator-valued Khintchine and Wick estimates belong to the classical noncommutative probability literature; see, for example, Haagerup–Pisier [2] and Buchholz [3]. We credit this phenomenon as a standard tool. The proof above supplies the exact Loewner form used here rather than inferring it from a trace-moment estimate.

3 Directions universal over all matrix dimensions

Lemma 3.1. *For the parameters in (1.2), there are vectors $g_{s,i} \in \mathbb{R}^m$, $1 \leq s \leq S$, $1 \leq i \leq N$, such that $u_{s,i} = g_{s,i}/\sqrt{m}$ satisfy*

$$\frac{1}{SN} \sum_{s,i} \left(\sum_j u_{s,ij} X_j \right)^{2q} \leq (4q/m)^q I \quad (3.1)$$

for every selfadjoint tuple of every finite dimension with $\sum_j X_j^2 \leq I$. They also satisfy, for every chart s ,

$$\sqrt{m}/2 \leq |g_{s,i}| \leq 2\sqrt{m}, \quad \frac{1}{2}I \leq N^{-1} \sum_i g_{s,i} g_{s,i}^T \leq 2I, \quad (3.2)$$

$$\sup_{|w| \leq m^{42}} \left| N^{-1} \sum_i g_{s,i} \sin \langle g_{s,i}, w \rangle - w e^{-|w|^2/2} \right| \leq 2/\sqrt{m}. \quad (3.3)$$

The directions within each chart are pairwise nonparallel.

Proof. Sample all $SN = m^{10q+6}$ vectors independently from standard Gaussian measure. For an ordered word $\alpha = (j_1, \dots, j_{2q})$, truncate $G^\alpha = G_{j_1} \cdots G_{j_{2q}}$ on $|G| > 2\sqrt{m}$. Its absolute value is at most $B = (2\sqrt{m})^{2q}$. Gaussian radial tails give

$$|\mathbb{E}G^\alpha - \mathbb{E}[G^\alpha \mathbf{1}_{|G| \leq 2\sqrt{m}}]| \leq (\mathbb{E}|G|^{4q})^{1/2} \mathbb{P}(|G| > 2\sqrt{m})^{1/2} \leq (Cm)^q e^{-cm} < m^{-3q},$$

for all sufficiently large m , since $q = O(\log m)$. Hoeffding's bounded-variable inequality at tolerance $m^{-3q} \leq B$ shows that the empirical truncated moment differs from its expectation by more than m^{-3q} with probability at most

$$2 \exp \left(-c \frac{m^{10q+6} m^{-6q}}{16^q m^{2q}} \right) = 2 \exp(-cm^{2q+6}/16^q).$$

A union bound over m^{2q} ordered words still tends to zero. Also $SN \mathbb{P}(|G| \notin [\sqrt{m}/2, 2\sqrt{m}]) \rightarrow 0$, because $\log(SN) = O((\log m)^2)$. On the latter event the sampled truncated moments equal the untruncated ones. Hence all empirical moments differ from their exact Gaussian values by at most $2m^{-3q}$ simultaneously.

For any admissible tuple, $X_j^2 \leq \sum_k X_k^2 \leq I$, so $\|X_j\|_{\text{op}} \leq 1$ and the norm of each word X_α is at most one. The factor m^{-q} from $u = g/\sqrt{m}$ therefore makes the empirical-minus-Gaussian operator moment error at most

$$m^{-q} m^{2q} 2m^{-3q} = 2m^{-2q}.$$

The error is selfadjoint. Lemma 2.1 bounds the exact normalized Gaussian moment by $(2q-1)!!m^{-q}I$, and $(2q-1)!!m^{-q} + 2m^{-2q} \leq (4q/m)^q$ for large m . This proves (3.1). The event fixed only scalar coefficients before a matrix tuple or its dimension was chosen; this is the source of the universal quantifier.

For clarity, the additional geometric direction events are the first two parts of the source directions lemma [1]. Its probabilistic proof bounds the Fourier/covariance failures by

$$\exp[-cm^4 + O(m \log m) + \log S]$$

and the norm failures by $SN e^{-cm}$. These still tend to zero for our $\log S = O((\log m)^2)$. The Fourier net has radius m^{-3} on a ball of radius m^{42} ; a $1/8$ sphere net controls vector and covariance norms. The truncated empirical Fourier function is $4m$ -Lipschitz and its exact Gaussian counterpart is 2-Lipschitz, which gives the displayed uniform error after net interpolation. Nonparallelism is almost sure. Intersect these events with the moment event and fix one realization. The source's scalar majority-good-chart assertion is not used. $\square$

Matching scalar polynomial moments is a standard design principle; exact averaging sets and spherical designs provide classical context [4]. We need only the elementary approximate Gaussian matching just proved, not a new design existence theorem or a sphere-design bound.

Set $(J_s \theta)_i = \langle u_{s,i}, \theta \rangle$ and $J_s^+ = (J_s^T J_s)^{-1} J_s^T$. The covariance conclusion gives

$$J_s^+ J_s = I, \quad \|J_s^+\|_{\text{op}} \leq 2\sqrt{m/N}, \quad |J_s^+ e_i| \leq 4m/N. \quad (3.4)$$

Here the last estimate uses $|u_{s,i}| \leq 2$ and $\|(J_s^T J_s)^{-1}\|_{\text{op}} \leq 2m/N$.

4 Spectral chart selection

We reserve M_s for the positive matrix controlling a chart and P_s for its spectral selector. Geometric chart complements will be R_s , and their extended Hilbert features will be $\mathcal{P}_s$.

Lemma 4.1. *Suppose the directions satisfy (3.1). Let $A_j, A_{s,j}$ and $b_{s,i}$ be selfadjoint matrices such that*

$$A_{s,j} = \sum_i u_{s,ij} b_{s,i}, \quad \|b_{s,i}\|_1 \leq t, \quad \sum_j \|A_{s,j} - A_j\|_1 \leq e$$

for every s , where $t, e \geq 0$. Then

$$\mathbb{E}_g \left\| \sum_j g_j A_j \right\|_1 \leq 2\sqrt{8 \exp(1)} N t \sqrt{q/m} + 2e. \quad (4.1)$$

The bound is independent of the matrix dimension.

Proof. Put $A(g) = \sum_j g_j A_j$, $U(g) = \text{sign}(A(g))$ with $\text{sign}(0) = 0$, and $X_j = \mathbb{E}[g_j U(g)]$. These are selfadjoint. For each vector v , Bessel's inequality in $L_2(\gamma_m; \mathbb{C}^D)$ gives

$$\sum_j |X_j v|^2 \leq \mathbb{E}|U(g)v|^2 \leq |v|^2.$$

Thus $\sum_j X_j^2 \leq I$. The essential positive pairing is

$$Y := \sum_j A_j X_j = \mathbb{E}[A(g) \text{sign}(A(g))] = \mathbb{E}|A(g)| \geq 0, \quad \text{Tr } Y = \mathbb{E}\|A(g)\|_1. \quad (4.2)$$

No positivity is asserted for the individual products $A_j X_j$.

Set

$$Z_{s,i} = \sum_j u_{s,ij} X_j, \quad M_s = \left(\sum_i Z_{s,i}^{2q} \right)^{1/q}.$$

Operator concavity of $T \mapsto T^{1/q}$ and (3.1) imply

$$S^{-1} \sum_s M_s \leq \left(S^{-1} \sum_{s,i} Z_{s,i}^{2q} \right)^{1/q} \leq N^{1/q} (4q/m) I \leq 4 \exp(1) (q/m) I.$$

Let $h = 8 \exp(1) q/m$ and $P_s = \mathbf{1}_{[0,h]}(M_s)$. Spectral calculus on each M_s gives $I - P_s \leq M_s/h$, whence

$$\sum_s P_s \geq (S/2) I. \quad (4.3)$$

Also $Z_{s,i}^{2q} \leq M_s^q$. Operator monotonicity of $T^{1/q}$ gives $Z_{s,i}^2 \leq M_s$, and hence

$$\|Z_{s,i}P_s\|_{\text{op}}^2 = \|P_sZ_{s,i}P_s\|_{\text{op}} \leq h. \quad (4.4)$$

For completeness, both fractional-power properties used here follow, for $0 < \alpha < 1$, from

$$T^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty T(T+sI)^{-1} s^{\alpha-1} ds.$$

The inverse reverses Loewner order and is operator convex. Therefore $I - s(T+sI)^{-1}$ is operator monotone and concave; integration preserves these properties. Continuity handles zero eigenvalues, and $\alpha = 1$ is immediate. Different M_s are never diagonalized simultaneously.

Since $Y \geq 0$, (4.3) can be paired with Y under the trace:

$$\begin{aligned} (S/2) \text{Tr } Y &\leq \sum_s \text{Tr}(YP_s) \\ &= \sum_{s,i} \text{Tr}(b_{s,i}Z_{s,i}P_s) + \sum_{s,j} \text{Tr}((A_j - A_{s,j})X_jP_s) \\ &\leq SNt\sqrt{h} + Se. \end{aligned}$$

Here $\text{Tr}(YP_s) = \text{Tr}(Y^{1/2}P_sY^{1/2})$ is real and nonnegative; the expanded summands can be nonreal, so the final line bounds their absolute values by trace duality. We use (4.4) and $\|X_jP_s\|_{\text{op}} \leq 1$. Division by $S/2$ and (4.2) prove (4.1). $\square$

5 The geometric construction with the larger chart family

This section uses the explicit operations in the pinned source [1], with every changed parameter tracked. We do not invoke its fixed- $S = m^3$ metric proposition unchanged.

5.1 Common kernel and exact chart extensions

Let $E(t) = (\cos t, \sin t)$ and let γ_m be Gaussian measure. Use the real Hilbert sum

$$\mathcal{H} = L_2([a, b] \times \mathbb{R}^m, d\sigma d\gamma_m; \mathbb{R}^2) \oplus \bigoplus_{s,i} L_2([a, b], d\sigma/\sigma; \mathbb{R}^2).$$

Define $p(\theta)$ to have first component $m^{1/4}E(\langle g, \theta \rangle/\sigma)$ and component (s, i) equal to $\sqrt{\varepsilon}E(\langle g_{s,i}, \theta \rangle/\sigma)$. These are exactly the source common-feature formulas, with our S and ε . They give

$$\begin{aligned} \langle p(\theta), p(\eta) \rangle &= \sqrt{m} \int_a^b e^{-|\eta-\theta|^2/(2\sigma^2)} d\sigma \\ &\quad + \varepsilon \sum_{s,i} \int_a^b \cos(\langle g_{s,i}, \eta - \theta \rangle/\sigma) \frac{d\sigma}{\sigma}. \end{aligned} \quad (5.1)$$

For $|\theta|, |\eta| \leq 3\sqrt{m}$, put $c_* = 2 \int_0^\infty (1 - e^{-1/(2t^2)}) dt > 0$. Scaling the Gaussian integral and bounding the omitted ends show that

$$|p(\theta) - p(\eta)|^2 - c_*\sqrt{m}|\theta - \eta| \leq 2\sqrt{m}a + 36m^{3/2}/b + 4\varepsilon SN \log(b/a) \leq m^{-37}. \quad (5.2)$$

In particular, $\varepsilon SN = m^{-194}$ despite the larger chart family. For $j = 0, 1, 2$, the squared derivative bounds from the Gaussian and auxiliary blocks are, respectively,

$$\sqrt{m} b a^{-2j} \mathbb{E}|G|^{2j}, \quad \varepsilon SN \log(b/a) (2\sqrt{m}/a)^{2j}.$$

Thus $|p|, \|Dp\|_{\text{op}}, \|D^2p\|_{\text{op}} \leq m^{110}$, with no $\sqrt{S}$ loss.

Let $B_s^0(\theta)e_i$ be supported in block (s, i) and equal there to

$$(m/N)\varepsilon^{-1/2}E'(\langle g_{s,i}, \theta \rangle / \sigma).$$

Define

$$B_s = B_s^0(I - J_s J_s^+) + Dp J_s^+, \quad R_s = I - J_s J_s^+. \quad (5.3)$$

Then $B_s J_s = Dp$ exactly. This is the source chart-derivative construction, renamed to avoid collision with the matrices of Lemma 4.1. Its profile remains

$$|\langle B_s(\theta)e_i, p(\eta) \rangle| \leq C(m/N) \log m \quad (|\theta|, |\eta| \leq 3\sqrt{m}). \quad (5.4)$$

Here is the explicit parameter check. In the main profile the factors $\sqrt{\varepsilon}$ and $\varepsilon^{-1/2}$ cancel, leaving $(m/N) \int_a^b \sin(\langle g_{s,i}, \eta - \theta \rangle / \sigma) d\sigma / \sigma$. The residual $Dp - B_s^0 J_s$ pairs with $p(\eta)$ through the Fourier error (3.3), because $|\eta - \theta| / \sigma \leq 6\sqrt{m}/a < m^{42}$. Its main part is at most $2 \log(b/a)$, and its auxiliary gradient is at most $2\varepsilon SN \sqrt{m}/a \leq 1$. Multiplying the residual by $J_s^+ e_i$ and using (3.4) proves (5.4).

The columns of B_s^0 have disjoint Hilbert supports. Even the looser bound by $\sqrt{N}$ times the largest column norm gives

$$\|B_s^0\|_{\text{op}} \leq \sqrt{N}(m/N)\varepsilon^{-1/2}\sqrt{\log(b/a)}, \quad \|DB_s^0\|_{\text{op}} \leq \sqrt{N}(m/N)\varepsilon^{-1/2}\sqrt{\log(b/a)} 2\sqrt{m}/a.$$

Both acquire only a factor $\sqrt{S}$ relative to the original powers of m . Moreover

$$DB_s = DB_s^0(I - J_s J_s^+) + D^2p J_s^+.$$

It follows that $\|B_s\|_{\text{op}}, \|DB_s\|_{\text{op}} \leq \sqrt{S}m^{300}$. Thus a common bound for all five strong norms is $\sqrt{S}m^{300}$, while the first three still satisfy the sharper m^{110} bound.

5.2 Rounding, angles and repair

Let $U = (-2, 2)^m$, and away from the threshold hyperplanes define

$$x_s(\theta)_i = \tau \lfloor \langle u_{s,i}, \theta \rangle / \tau \rfloor, \quad v_s(\theta) = (s, x_s(\theta)), \quad V = \{v_s(\theta) : \theta \in U \text{ off those hyperplanes}\}.$$

Set

$$\mathcal{P}_s(x) = p(J_s^+ x) + B_s(J_s^+ x)R_s x, \quad \mathcal{P}_v = \mathcal{P}_s(x) \quad (v = (s, x)).$$

Rounding gives

$$|x - J_s \theta| \leq \sqrt{N}\tau, \quad |J_s^+ x - \theta| \leq 2\sqrt{m}\tau, \quad |R_s x| \leq \sqrt{N}\tau. \quad (5.5)$$

Consequently, the feature error satisfies

$$|\mathcal{P}_{v_s(\theta)} - p(\theta)| \leq \sqrt{S}m^{305}\tau.$$

The resulting squared-distance error is at most $CS^{1/2}m^{415}\tau \leq Sm^{610}\tau$. By (1.2), $Sm^{610}\tau = m^{-1390}/S \ll r$, so

$$|\mathcal{P}_{v_s(\theta)} - \mathcal{P}_{v_t(\eta)}|^2 - c_*\sqrt{m}|\theta - \eta| \leq r/10. \quad (5.6)$$

The three-distance polarization identity and the Euclidean triangle inequality then give every angle a lower bound $-r$.

For same-chart labels x, x' , every point ξ on their segment satisfies the last two bounds of (5.5), with $|J_s^+ \xi| \leq 3\sqrt{m}$. Differentiation gives the exact identity

$$D\mathcal{P}_s(\xi)v = B_s(J_s^+ \xi)v + DB_s(J_s^+ \xi)[J_s^+ v]R_s \xi. \quad (5.7)$$

The second term has operator norm at most $2\sqrt{S}m^{300.5}\tau$. When paired with $\mathcal{P}_y$, it contributes at most $C\sqrt{S}m^{411}\tau$. Replacing $\mathcal{P}_y$ by its common feature in the first term costs at most $CSm^{605}\tau$. These are distinct error sources; both are bounded by $CSm^{610}\tau \ll (m/N)\log m$. Equation (5.4), applied also to $\mathcal{P}_z$, and integration along the segment therefore give, with $L = C_0 \log m$,

$$|\langle \mathcal{P}_v - \mathcal{P}_z, \mathcal{P}_y - \mathcal{P}_z \rangle| \leq L(m/N)\|x - x'\|_1 \quad (v = (s, x), z = (s, x')). \quad (5.8)$$

Set $\delta(v, w) = L(m/N)\|x - x'\|_1$ within a chart and $\delta = \infty$ between charts. The source triangle-repair lemma applies to any finite chart partition: if the angles are at least $-\min\{r, \delta(v, z), \delta(y, z)\}$, it appends Hilbert vectors q_v with

$$|q_v - q_w|^2 = 8 \int_0^r (1 - e^{-\delta(v, w)/\ell}) d\ell \leq 8r \quad (5.9)$$

such that $d(v, w) = |\mathcal{P}_v - \mathcal{P}_w|^2 + |q_v - q_w|^2$ obeys all triangle inequalities. The global bound and (5.8) give exactly its hypothesis. Its positive kernel is $4 \int_0^r e^{-\delta(v, w)/\ell} d\ell$, block diagonal across charts; this is the credited Schoenberg-kernel repair, not a new operation. Distinct same-chart labels have $\delta > 0$, and between charts (5.9) equals $8r$. Thus d is already a genuine negative-type metric on the label set.

Taking $y = v$ in (5.8) bounds the original squared feature distance by δ . For $0 < \delta \leq r$, the repair cost is at most $8\delta(1 + \log(r/\delta))$. Therefore adjacent labels satisfy

$$\|x - x'\|_1 = \tau \implies d((s, x), (s, x')) \leq C\tau(m/N)(\log m)^3, \quad (5.10)$$

because $\log(r/(L(m/N)\tau)) = 1997 \log m + 2 \log S - \log(C_0 \log m) = O((\log m)^2)$. The macro bound is

$$|d(v_s(\theta), v_t(\eta)) - c_*\sqrt{m}|\theta - \eta|| \leq Cr. \quad (5.11)$$

At most $H_0 = N(1 + 8\sqrt{m}/\tau)$ threshold hyperplanes meet U in one chart. The arrangement bound, proved by induction on the number of hyperplanes, gives at most $\sum_{j=0}^m \binom{H_0}{j} \leq (H_0 + 1)^m$ regions. Each label is constant on a region, so

$$|V| \leq S(H_0 + 1)^m \leq \exp(Cm(\log m)^2), \quad \text{diam}(V, d) \leq Cm. \quad (5.12)$$

Finally let $T = (-1, 1)^m$. For independent uniform $\theta, \eta \in T$, $\mathbb{E}|\theta - \eta|^2 = 2m/3$ and $|\theta - \eta| \leq 2\sqrt{m}$, whence $\mathbb{E}|\theta - \eta| \geq \sqrt{m}/3$. It follows that

$$\mathbb{E}_{\theta, \eta \in T} d(v_1(\theta), v_1(\eta)) \geq cm. \quad (5.13)$$

6 Trace-norm contraction on the label set

First let $F : V \rightarrow M_D(\mathbb{C})$ have selfadjoint values and be nonexpanding for d . Set $f_s(\theta) = F(v_s(\theta))$ almost everywhere on U and extend by zero outside U , assigning zero also on threshold hyperplanes. Choose a fixed even nonnegative $\rho \in C_c^\infty((-1, 1))$ of integral one and let

$$\kappa_\lambda(z) = \prod_{j=1}^m \lambda^{-1} \rho(z_j/\lambda), \quad H_s = f_s * \kappa_\lambda.$$

These are smooth finite-dimensional matrix-valued functions. For $\theta \in \overline{T}$, the convolution kernel is supported inside U . The macro bound (5.11) implies

$$\begin{aligned} \|H_s(\theta) - f_s(\theta)\|_1 &\leq C(m\lambda + r) && \text{for almost every } \theta \in T, \\ \|\partial_j H_s(\theta) - \partial_j H_1(\theta)\|_1 &\leq Cr/\lambda && (\theta \in T). \end{aligned} \quad (6.1)$$

Indeed, kernel displacements have Euclidean norm at most $\sqrt{m}\lambda$, and $\|f_s - f_1\|_1 \leq Cr$ pointwise inside U . Differentiating the kernel uses $\int |\partial_j \kappa_\lambda| = \lambda^{-1} \|\rho'\|_1$. These are norm inequalities and require no scalar-coordinate decomposition.

6.1 The pointwise interface coefficient bound

Fix s and write $u_i = u_{s,i}$. Across the interface $Q_{i,k} = \{x : \langle u_i, x \rangle = k\tau\}$, let $\Delta_{i,k}(x)$ be the jump of f_s toward increasing $\langle u_i, x \rangle$. It exists almost everywhere on the facet. Other interfaces intersect it in codimension at least two, by nonparallelism. Adjacent labels differ by τe_i , so

$$\|\Delta_{i,k}(x)\|_1 \leq C\tau(m/N)(\log m)^3.$$

Integration by parts on the polyhedral cells gives, entrywise, the distributional gradient inside U . After convolution it is the classical identity, at every $\theta \in T$,

$$\partial_j H_s(\theta) = \sum_i u_{s,ij} b_{s,i}(\theta), \quad b_{s,i}(\theta) = \frac{1}{|u_i|} \sum_k \int_{Q_{i,k} \cap U} \kappa_\lambda(\theta - x) \Delta_{i,k}(x) d\mathcal{H}^{m-1}(x). \quad (6.3)$$

The cell interiors contribute zero; paired facet terms give the signed jump and its unit normal. The testing kernel vanishes near ∂U , so there is no outer boundary term.

Let h_i be the density of $\langle u_i, Z \rangle$ when Z has density $\kappa_\lambda(\theta - \cdot)$. Coarea identifies

$$h_i(t) = |u_i|^{-1} \int_{\langle u_i, x \rangle = t} \kappa_\lambda(\theta - x) d\mathcal{H}^{m-1}(x).$$

As $|u_i| \geq 1/2$, some coordinate has modulus at least $1/(2\sqrt{m})$. Conditioning on all other coordinates shows that h_i is a mixture of translated or reflected one-dimensional kernels at scale at least $\lambda/(2\sqrt{m})$. Hence $\int |h'_i| \leq C\sqrt{m}/\lambda$. Comparing values and integrals on each interval of length τ gives

$$\tau \sum_k h_i(k\tau) \leq 1 + \tau \int |h'_i| \leq 1 + C\tau\sqrt{m}/\lambda \leq 2.$$

The nonnegative kernel in (6.3) now proves the pointwise, rather than merely integrated, bound

$$\|b_{s,i}(\theta)\|_1 \leq C(m/N)(\log m)^3. \quad (6.4)$$

This is the source coarea calculation in trace norm; the matrices $b_{s,i}$ need not be positive.

Apply Lemma 4.1 with $A_j = \partial_j H_1(\theta)$, $A_{s,j} = \partial_j H_s(\theta)$, $t = C(m/N)(\log m)^3$ and $e = Cmr/\lambda = Cm^{-5}$. Since $q = O(\log m)$, it yields

$$\mathbb{E}_g \left\| \sum_j g_j \partial_j H_1(\theta) \right\|_1 \leq C\sqrt{m}(\log m)^{7/2} \quad (\theta \in T). \quad (6.5)$$

6.2 A Banach-valued first-moment cube inequality

Lemma 6.1. *For a finite-dimensional normed space B and a C^1 map H on a neighborhood of $\bar{T}$,*

$$\mathbb{E}_{\theta, \eta \in T} \|H(\theta) - H(\eta)\|_B \leq \sqrt{\pi/2} \mathbb{E}_{\theta \in T} \mathbb{E}_g \left\| \sum_j g_j \partial_j H(\theta) \right\|_B.$$

Proof. Let G, G' be independent standard Gaussian vectors and, for $0 \leq t \leq \pi/2$, put $G(t) = \cos t G + \sin t G'$ and $W(t) = -\sin t G + \cos t G'$. At fixed t , these are independent standard Gaussian vectors. With Φ, ϕ the Gaussian distribution function and density, set $\theta_j(t) = 2\Phi(G_j(t)) - 1$. The endpoints are independent uniform points of T , and the derivative is

$$\frac{d}{dt} H(\theta(t)) = \sum_j 2\phi(G_j(t)) W_j(t) \partial_j H(\theta(t)).$$

The normalized coefficients $2\phi(G_j(t))/\sqrt{2/\pi}$ belong to $[0, 1]$. For any vectors v_j and $0 \leq a_j \leq 1$, adding independent noise $\sqrt{1 - a_j^2} g'_j$ and applying Jensen proves $\mathbb{E} \|\sum a_j g_j v_j\| \leq \mathbb{E} \|\sum g_j v_j\|$. Condition on $G(t)$, use this contraction inequality, and integrate the norm of the derivative over $[0, \pi/2]$. Since $\theta(t)$ is uniform for every t , the constant is $(\pi/2)\sqrt{2/\pi} = \sqrt{\pi/2}$. The derivatives are bounded on the compact cube and Gaussian first moments are finite, so the integrations are justified. This is the standard Gaussian rotation method [5], in the norm-valued form needed here. $\square$

Combining Lemma 6.1, (6.5) and (6.1), and using $m\lambda = m^{-1}$, proves

$$\mathbb{E}_{\theta, \eta \in T} \|F(v_1(\theta)) - F(v_1(\eta))\|_1 \leq C\sqrt{m}(\log m)^{7/2}. \quad (6.6)$$

For arbitrary complex matrix values use the real-linear Hermitian dilation

$$\mathcal{D}(B) = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \quad \|\mathcal{D}(B)\|_1 = 2\|B\|_1.$$

The map $\mathcal{D}(F)/2$ is selfadjoint and preserves every trace-norm pair distance. Thus (6.6) holds without a selfadjointness assumption.

7 Uniform copies, arbitrary Hilbert spaces and the exponent

Let $w_v = \mathbb{P}(v_1(\theta) = v)$ for uniform $\theta \in T$, retaining all vertices including those of weight zero. Set

$$M = m^m + m^2|V|, \quad k_v = 1 + \lfloor Mw_v \rfloor, \quad n = \sum_v k_v, \quad \mu_v = k_v/n.$$

Writing $k_v = Mw_v + e_v$ with $0 < e_v \leq 1$, we obtain $M \leq n \leq M + |V|$ and

$$\text{TV}(\mu, w) = \frac{1}{2} \sum_v |\mu_v - w_v| \leq |V|/M \leq m^{-2}.$$

The product laws differ in total variation by at most $2m^{-2}$. Both the metric and every nonexpanding image distance are at most Cm . Consequently (6.6) and (5.13) imply, for every finite-matrix-valued nonexpanding F ,

$$\mathbb{E}_{v, w \sim \mu} \|F(v) - F(w)\|_1 \leq C\sqrt{m}(\log m)^{7/2}, \quad \mathbb{E}_{v, w \sim \mu} d(v, w) \geq cm. \quad (7.1)$$

The change in either expectation is at most $2C/m$.

Make k_v copies of each vertex. Write d_0 for the pulled-back pseudometric and let $d_{\min} > 0$ be the least distance between distinct original labels. Choose $0 < \zeta \leq \min(1, d_{\min})/m^2$ and define

$$d'(x, y) = d_0(x, y) + \zeta \mathbf{1}_{x \neq y}.$$

The added equilateral metric is squared Hilbert distance, for example using $\sqrt{\zeta/2}$ times distinct standard unit vectors. Therefore d' is a genuine negative-type metric, and its uniform metric average is at least the μ -average in (7.1).

Here is the full reduction for a map F on the copies with values in $S_1(H)$, in an order that makes the quantifiers explicit. First collect the ranges of the finitely many $F(x)$ and $F(x)^*$ into a separable closed subspace $K \subseteq H$. This is possible because trace-class operators are compact; their ranges are separable. Moreover $F(x) = P_K F(x) P_K$. Choose finite-rank orthogonal projections E_ℓ increasing strongly to P_K and put $F_\ell(x) = E_\ell F(x) E_\ell$. Compression is trace-norm contractive, so every F_ℓ is nonexpanding into a finite matrix algebra.

Choose one copy x_v of each label and put $G_\ell(v) = F_\ell(x_v)$. For distinct labels,

$$d'(x_v, x_w) = d(v, w) + \zeta \leq (1 + m^{-2})d(v, w).$$

Thus $G_\ell/(1 + m^{-2})$ is nonexpanding on the label set. Hermitian dilation reduces this finite matrix map to the selfadjoint contraction already proved, so (7.1) applies. Each copy is at distance ζ from its representative. It follows that

$$\mathbb{E}_{x,y} \text{ uniform} \|F_\ell(x) - F_\ell(y)\|_1 \leq (1 + m^{-2})C\sqrt{m}(\log m)^{7/2} + 2\zeta. \quad (7.2)$$

Finally $E_\ell A E_\ell \rightarrow A$ in trace norm for every trace-class A supported on K . To see this without a uniform operator-norm convergence assertion, approximate A in trace norm by a finite-rank operator. On its finitely many rank-one terms strong convergence of the projections gives trace-norm convergence; compression contracts the remaining trace-norm error. Passing to the limit in the finite pair sum in (7.2) establishes it for F . This argument also covers nonseparable H . The reduction order was compression, representatives, dilation, the finite estimate, and then the trace-norm limit.

By (5.12),

$$m^m \leq n \leq \exp(Cm(\log m)^2).$$

For sufficiently large m , this implies $\log \log n \geq \log m$ and $\sqrt{\log n} \leq C\sqrt{m} \log m$. The ratio of the metric average to the image average is therefore at least

$$c \frac{\sqrt{m}}{(\log m)^{7/2}} \geq c' \frac{\sqrt{\log n}}{(\log \log n)^{9/2}}.$$

Taking one construction for each sufficiently large m proves Theorem 1.1. All metric choices precede the choice of H or F .

8 Comparison with other trace-class operations

The precise claim is new to this project, with no antecedent found in a bounded search. The individual tools above are standard; no global priority claim follows from the search or from the shortness of the matrix bridge. We record the operations that distinguish the statement, rather than claiming that alternative methods cannot ever apply.

8.1 Dimension reduction, snowflakes and quantum expansion

Naor–Pisier–Schechtman [6] study dimension reduction for subsets already embedded in trace class, including embeddings into arbitrary low-dimensional subspaces. Their discussion explains why scalar coordinate sampling does not automatically have a matricial version. It does not provide the all-target-dimensions obstruction for our negative-type source family. Naor–Young [7], Question 13 and its discussion, concern Heisenberg groups and trace-class nonembedding. Our metrics are different; no conclusion about those groups is proved here.

Kush–Nikolov–Tang [8], Theorem 34 of the cited preprint, embed the $p/2$ -snowflake of Schatten p into Hilbert space with constant average distortion for $1 \leq p \leq 2$. At $p = 1$ the source distance has been replaced by its square root. Composing with a trace-class map of our negative-type metric therefore controls a Hilbert map for $\sqrt{d}$, which is already Euclidean. This does not supply the needed contraction of first averages for d itself.

Escudero Gutiérrez–Muguruza [14] compare expansion notions for compressed blocks of bistochastic matrix tuples. Their Section 3.2 distinguishes that operation from metric Poincaré inequalities for arbitrary maps. We do not derive an arbitrary-map inequality from channel expansion, and we do not claim to settle their general Question 2.

8.2 Talagrand type and nonlinear cube quotients

Luo–Wang–Zhou [9], Theorem 1.1, give a noncommutative L_1 cube inequality with a sum of coordinate Orlicz increments. Its input is different from the simultaneous-chart representation controlling $\mathbb{E}\|\sum g_j A_j\|_1$ in Lemma 4.1. We do not claim that no dimension-free trace-class functional inequalities were known.

Cordero-Erausquin–Eskenazis [10], Theorem 13, combined with [9], gives a genuine alternative route through quotients of the cube with small geometric influences. Although their displayed theorem is a bi-Lipschitz lower bound, equation (123) in its proof is already a pair-average inequality, before any co-Lipschitz assumption is inserted in (124). For a nonexpanding quotient map, the coordinate increment is bounded by one half of the corresponding boundary indicator, as in (125). Hence their proof bounds the pair average under the pushforward of uniform cube measure. It would be incorrect to dismiss this route because the theorem is phrased in bi-Lipschitz terms.

For precision, write $\mathfrak{t}$ for the Talagrand-type constant (to distinguish it from our mesh τ), σ_i for the coordinate boundary probabilities, and a_1 for the quotient metric average. For $\psi(t) = t/\log(e+t)$, the resulting lower bound is

$$\frac{2a_1}{\mathfrak{t} \sum_{i:\sigma_i>0} 1/\psi^{-1}(1/\sigma_i)}.$$

A random coordinate path gives $a_1 \leq \frac{1}{2} \sum_i \sigma_i$: each coordinate is flipped with probability $1/2$, and the intermediate point is uniform. Also $\psi^{-1}(u) \leq 2u \log(e+u)$, since $e + 2u \log(e+u) \leq (e+u)^2$. Thus the displayed lower bound is at most

$$2\mathfrak{t}^{-1} \max_{i:\sigma_i>0} \log(e + 1/\sigma_i). \quad (8.1)$$

The trivial one-class quotient can be discarded. This is an influence-logarithmic cap, not an $O(\log \log \mathcal{N})$ cap in the number $\mathcal{N}$ of quotient classes; no relation forcing the latter is available. Theorem 42 of [10] supplies the same Talagrand-type property for L_1 as [9] supplies for S_1 , with universal constants (a common upper bound can be used for both).

On our source representation the direct coordinate-sum estimate is

$$\sum_{j=1}^m \mathbb{E} \|\partial_j H_1\|_1 \leq C m N m^{-1/2} \frac{m}{N} (\log m)^3 = C m^{3/2} (\log m)^3,$$

whereas the proof needs $C\sqrt{m}(\log m)^{7/2}$. The missing gain is polynomial Gaussian cancellation, not merely a change of constants. An influence-logarithmic route could supply such a gain only after an appropriate very-small-influence representation is constructed. The compared inputs supply no genuine negative-type quotient family with that influence control, the required average scale, and uniformization when classes have unequal sizes. They do not identify the multichart metric with such a quotient. This is a missing construction in the read inputs, not an impossibility theorem about Talagrand-type arguments.

8.3 Sharp metric cotype and the usual grids

Jiao–Luo–Wu [11], posted October 15, 2026, before this candidate, prove sharp metric cotype 2 for S_1 (Theorem 1.3). In their notation, for every even $k \geq 4$ and $f : \mathbb{Z}_k^d \rightarrow S_1$,

$$\sum_{j=1}^d \mathbb{E}_x \|f(x + (k/2)e_j) - f(x)\|_1^2 \leq (3\sqrt{70}k + 6\sqrt{d})^2 \mathbb{E}_{x,\epsilon} \|f(x + \epsilon) - f(x)\|_1^2,$$

where ϵ is uniform on $\{-1, 0, 1\}^d$. Thus the scale can be $k \asymp \sqrt{d}$. Their paper also constructs nonlinear matrix smoothing; its existence is fully credited.

The general-metric cardinality bound below is classical. As recorded in [13], Section 1.3, Bourgain’s 1987 grid theorem gives distortion $\gtrsim_Y d^{1/q}$ for embeddings of $[[d^{1+1/q}]]_\infty^d$ into a Banach space Y of Rademacher cotype q ; GMN’s Theorem 1.2 also yields this bound. The same section credits Tomczak-Jaegermann (1974) for cotype 2 of S_1 . Here $[m]_\infty^d$ denotes $\{0, \dots, m\}^d$ with the sup metric. Thus, for $q = 2$, $\mathcal{N} = (\lceil d^{3/2} \rceil + 1)^d$ and $\log \mathcal{N} = (3/2 + o(1))d \log d$ give distortion at least $c\sqrt{\log \mathcal{N} / \log \log \mathcal{N}}$. These two classical attributions are through GMN’s explicit statements; the original Bourgain and Tomczak-Jaegermann bodies were not read here.

The JLW/MN chain recovers this cardinality order with grid side $O(\sqrt{d})$, instead of order $d^{3/2}$. The grid nonembedding deduction inserts a co-Lipschitz bound for long coordinate jumps into the cotype inequality; see [12], proof of Theorem 1.12. JLW’s Corollary 6.1 has constant 32 at even scale $k \geq \max(4, \sqrt{d})$. For a nonexpanding map of the sup-metric torus $\mathbb{Z}_k^d$ with inverse Lipschitz constant D , its left side is at least $d(k/(2D))^2$, while its right side is at most $(32k)^2$. Consequently $D \geq \sqrt{d}/64$. This is the proof of [12], Theorem 1.12, with the new sharp inequality substituted. The third assertion of [12], Lemma 6.12, embeds the torus with constant distortion into a sup-grid of side $O(k)$ and dimension $O(d)$. Taking $k \asymp \sqrt{d}$ gives such grids with $\mathcal{N}$ points and trace-class bi-Lipschitz distortion at least

$$c\sqrt{\frac{\log \mathcal{N}}{\log \log \mathcal{N}}}.$$

This is the same order in $\mathcal{N}$ as the classical bound, not an improvement of that order. The sharp scale improves the grid side, addressing the S_1 case of [13], Question 3. The classical bound is already stronger than our bi-Lipschitz corollary for general metrics. Any possible novelty of that corollary is therefore confined to the negative-type specialization, under the bounded-search qualification above. The grid consequence is not the first-average negative-type statement of Theorem 1.1, as the following exact checks show.

Proposition 8.1. *For $G = \{0, \dots, k\}^d$ with $k, d \geq 1$ and the sup metric, there is a nonexpanding map to $S_1(\mathbb{C})$ preserving at least one third of the uniform first average. If $d \geq 3$ and $k \geq 2$, this grid metric is not of negative type.*

Proof. More generally choose a rank-one orthogonal projection P in any nonzero Hilbert space and set $F(x) = x_1 P$. Then $\|F(x) - F(y)\|_1 = |x_1 - y_1| \leq \|x - y\|_\infty$. For independent uniform $U, V \in \{0, \dots, k\}$,

$$\mathbb{E}|U - V| = \frac{2}{(k+1)^2} \sum_{h=1}^k h(k+1-h) = \frac{k(k+2)}{3(k+1)} \geq k/3.$$

Since $\mathbb{E}\|x - y\|_\infty \leq k$, the first assertion follows.

For the second assertion take, in the side-two three-dimensional grid,

$$A = \{(2, 1, 0), (0, 1, 0), (0, 1, 1)\}, \quad B = \{(1, 0, 1), (1, 0, 0), (1, 2, 0)\}.$$

Assign coefficients $+1$ to A and -1 to B . Their sum is zero. Each unordered within-set distance sum is 5, and all nine cross distances equal 1. Thus

$$\sum_{i,j} c_i c_j \|x_i - x_j\|_\infty = 2 \cdot 5 + 2 \cdot 5 - 2 \cdot 9 = 2 > 0.$$

If these distances were $|z_i - z_j|^2$, the same expression would equal $-2|\sum_i c_i z_i|^2 \leq 0$. This contradiction persists in every larger such grid by inclusion. $\square$

The proposition excludes the standard grid substitution, not every use of metric cotype. To use the cotype inequality for our uniform first average, one would need a torus map into the negative-type family whose single-coordinate antipodal pair law controls a fixed fraction of its uniform pair law, with sufficiently short local jumps. No such map is supplied by the compared inputs. Cotype itself is a lower-bound property of random sums, whereas Lemma 4.1 gives an upper bound on $\mathbb{E}\|\sum_j g_j A_j\|_1$.

There is also a concrete loss in the nonlinear smoothing alternative. Proposition 5.2 of [11] bounds the ℓ_2 sum of trace-norm partial derivatives by $6\sqrt{70}$ times the input oscillation. In S_1 , converting that quantity to a Gaussian average can lose $\sqrt{m}$: for $A_j = e_j e_j^*$,

$$\left(\sum_j \|A_j\|_1^2\right)^{1/2} = \sqrt{m}, \quad \mathbb{E}\|\sum_j g_j A_j\|_1 = m\sqrt{2/\pi}.$$

That is precisely the size of gain at issue. Moreover, the macro bound at smoothing scale λ gives input oscillation $O(m\lambda)$; a derivative estimate after rescaling and this ℓ_2 -to-Gaussian conversion again permits $O(m^{3/2})$. The nonlinear smoothing controls oscillation, not the simultaneous representation $A_{sj} = \sum_i u_{sij} b_{si}$ used by the positive spectral selection. These comparisons do not rule out future combinations of the methods. The Bessel step in [11], equation (4.5), and its Hermitian dilation and finite-rank approximation (Steps II–III) are parallel standard devices; they are not claimed as new here.

9 Scope of the accompanying formal support

The accompanying author-side Lean development uses Lean 4.34.1 and mathlib revision

d13f23b723b8a846827a245b89c10fc7d3f11612.

It contains 31 support declarations and 15 control declarations. For arbitrary finite complex positive matrices M_s it constructs actual spectral projections and proves their defect and compression bounds, the half-coverage conclusion from the summed budget, and the positive trace pairing without a commutation assumption. It also proves the per-chart budget consequence. The controls include noncommuting rank-one projections, necessity of positivity in the trace pairing, a zero-threshold failure for the abstract defect interface, and exact threshold endpoint behavior. These are general kernel-checked lemmas and hypothesis tests, not a numerical proof of the metric theorem.

The support does not formalize the Wick/register argument, Gaussian integration or finite direction construction; fractional powers and the Loewner–Heinz/concavity input; the conversion of compressed squares to operator norms; the Gaussian construction of Y or trace-norm duality; source geometry, coarea, copy uniformization, asymptotics or arbitrary- H compression. Its positive matrix and summed-budget hypotheses still have to be supplied by the mathematical proof above. The build audits all declaration types and allows only the standard axioms `propext`, `Classical.choice` and `Quot.sound`. The pinned dependency object cache is trusted, not rebuilt. A separate independent review passed this exact scoped support, with no required repairs. Its additional checks exercise actual coverage and trace interfaces on noncommuting data. In particular, the genuine zero-matrix spectral cutoff at threshold zero is I ; the abstract zero-threshold counterexample is not a counterexample to that cutoff. The chosen eigenbasis is used to construct the projection; independence of that choice or identification with continuous functional calculus is not formalized. Only the proved projection properties are used here. Neither the 46 author declarations nor the review constitute a formal proof of Theorem 1.1.

Reproduction. The accompanying packet README gives the PDF command

```
python3 -B build.py --output /var/tmp/NEW-PDF-BUILD
```

with a fresh output directory and an explicit Tectonic option if needed. The separate scoped formal packet `t117q-formal-v1/rebuild.py` provides the Lean entry point with `--output`, `--lean` and `--cache` arguments; its README specifies the pinned tools and trusted dependency cache. These builds do not replace the stated mathematical and formal-coverage limits.

A Exact source interfaces and reading limits

The source [1] is pinned at OpenAI/math revision

fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb.

Its path is

```
preprints/Near-square-root-logarithmic-integrality-gaps-for-uniform-sparsest-cut-September-24-2026/build/source/.
```

The complete preprint-tree inventory, not a truncated root-tree listing, identifies the inputs. Source truth is stipulated. The exact reused interfaces are:

- `directions.tex`: `lem:directions`, its norm, covariance and Fourier approximation parts, and `eq:pseudoinverse`. The original $S = m^3$ assertion is not used unchanged; Section 3 rechecks the probability bounds in `directions-proof.tex`. Scalar majority charts are unused.

- `kernel.tex`: `eq:kernel`, `ker:A-definition`, `ker:extension-identity`, `ker:profile`, `ker:strong-bounds`, `ker:P-definition` and `ker:DP-identity`. Section 5 reproduces the formulas and checks the new S, ε, τ dependence explicitly.
- `repair.tex`: `lem:repair`, including `rep:eq:hypothesis`, `rep:eq:distance` and `rep:eq:local`, stated for an arbitrary finite chart partition.
- `metric.tex`: the counting, diameter and average arguments behind `met:eq:macro`, `met:eq:adjacent`, `met:eq:size-diameter` and `met:eq:average`. Their modified bounds are proved here, not imported at the old mesh.
- `contraction.tex`: smoothing, derivative mismatch, distributional gradient, coarea coefficient formula, density variation and lattice-sum bound. Its scalar-coordinate good-chart step is replaced by Lemma 4.1; its Gaussian rotation is used in the norm-valued form proved in Lemma 6.1.
- `reduction.tex`: the standard finite replication idea. Section 7 proves the precise uniformization used here and adds an equilateral term to retain strict positivity.

The constant-factor-hardness companion was keyword-screened, not read through its operative proof, and is not a dependency. No assertion about that full proof is used. Tropp’s matrix-concentration text, accessed only in earlier browser passages without an authenticated local body, is orientation rather than evidence and is not needed here. The classical operator Khintchine and design references are credits to established methods, not assertions that their complete proofs were read in this manuscript task. The new lemmas needed here have their proofs above.

Primary comparisons are limited to the stated theorems and operative passages of [6, 7, 8, 9, 10, 11, 12, 13, 14]; no exhaustive bibliography or forward-citation search was performed. The full mathematical proof passed a separate proof review, and contribution round 1 classified the new bridge as substantial to the project while requiring the positioning implemented in Section 8. A separate round 2 contribution review retained that project-level classification, with the corrected influence cap, concrete smoothing loss and stronger general-metric grid consequence now recorded in Section 8. Neither review establishes literature priority or constitutes manuscript acceptance.

Acknowledgments and license. Parent author collaboration checked the register-contraction and selector interfaces and supplied the explicit grid-average and six-point certificates in Proposition 8.1. This assistance is author work, not independent review. Original prose is licensed CC BY 4.0 and original accompanying code MIT. Third-party manuscripts and their licenses are separate; private comparison bodies are not redistributed.

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