

The sharp minimum multiplicity in a staircase tensor square

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Abstract

Let $\rho_m = (m, m-1, \dots, 1)$ and $N_m = m(m+1)/2$. Assuming the pinned cyclic-polytabloid source theorems specified here, we prove that every irreducible complex representation of S_{N_m} , other than the trivial and sign representations, occurs in $S^{\rho_m} \otimes S^{\rho_m}$ with multiplicity at least $m-1$, for every $m \geq 2$. The bound is sharp at $(N_m-1, 1)$. The proof quantitatively evaluates an inherited recursion for a specified cyclic submodule. An asymmetric invariant has four exceptional shapes. Two nonexceptional predecessors handle diagrams with large first-four-row capacity; explicit rigged configurations give a uniform Kostka bound in the complementary range. Exact finite calculations supply the starts through order eight. The uniform estimate is presented as new to this project; new-to-literature priority is not claimed.

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Contents

1	Statement, notation, and dependencies	2
2	The cyclic source and the inherited recursion	2
3	Wide diagrams	4
4	Two predecessors at high capacity	5
5	A uniform low-capacity Kostka bound	6
5.1	The configuration convention	6
6	Closing the uniform induction	9
7	Exact finite starts and reproducibility	9
7.1	Algorithms and their mathematical contracts	10
7.2	Finite conclusions	10
7.3	Reproduction and limits	11
8	Conclusion and attribution	11

1 Statement, notation, and dependencies

All representations in this article are finite-dimensional complex representations. Tensor products of representations of one symmetric group carry its diagonal action. For partitions $\alpha, \beta, \lambda \vdash n$, write

$$g(\alpha, \beta, \lambda) = \dim \operatorname{Hom}_{S_n}(S^\lambda, S^\alpha \otimes S^\beta).$$

Partitions are padded by zero parts when necessary; λ^t denotes conjugation and $\ell(\lambda)$ denotes the number of positive parts. Put $\rho_m = (m, m-1, \dots, 1)$ and $N_m = m(m+1)/2$.

Theorem 1.1. *For every integer $m \geq 2$ and every $\lambda \vdash N_m$ with $\lambda \notin \{(N_m), (1^{N_m})\}$,*

$$g(\rho_m, \rho_m, \lambda) \geq m-1. \quad (1.1)$$

Moreover $g(\rho_m, \rho_m, (N_m-1, 1)) = m-1$.

The source theorems of OpenAI [1, 2] are used at the fixed revision

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Their truth is an explicit premise of this work. We do not reprove or audit them. Section 2 states exactly which module-level conclusions are used; ordinary Saxl positivity by itself would not suffice. The other non-elementary combinatorial input is the standard positive rigged-configuration formula for Kostka numbers, in the convention established in Section 5.1 from Kirillov–Schilling–Shimozono [4]. The finitely many starts are exact computations described in Section 7. Every remaining all-order estimate is proved below.

The band multiplicity recursion, including the rule of taking the maximum of separate quotient bounds, was developed in I19, Section 4 [8]. The dominance-module and Kostka refinement was developed in I21, Section 2.1 [9], using the source’s explicit maps. Neither is a new construction here. The residual result is the uniform evaluation of this recursion: a fixed exceptional set, a predecessor estimate, and a quantitative low-capacity Kostka bound. The high-capacity predecessor argument was supplied by parent author assistance and is incorporated in full in Section 4; that assistance is part of the author development, not an independent review.

Our claim concerns the ordinary tensor square. It does not give the multiplicity separately in its symmetric and exterior squares, in the cyclic submodule, over the integers, or in positive characteristic. In particular it does not resolve the earlier signed-square target of I19 and I21. The familiar corner formula supplies sharpness, not a new representation-theoretic input.

2 The cyclic source and the inherited recursion

Let

$$B_m = \{(i, j) \in \mathbb{Z}_{\geq 1}^2 : i + j \leq m+1\}, \quad G_m = S_{B_m}.$$

For an ordered block of r positions, alternate the tensor $e_1 \otimes \dots \otimes e_r$ without normalization. Taking the tensor product of these alternating tensors over rows of B_m gives v_R ; taking them over columns gives v_C . Rows are ordered by increasing column index and columns by increasing row index. With $E = \mathbb{C}^m$, these are tensors in $E^{\otimes B_m}$. The pinned source identifies both orbit spans with S^{ρ_m} and defines

$$W_m = \mathbb{C}[G_m](v_R \otimes v_C) \subseteq S^{\rho_m} \otimes S^{\rho_m}.$$

This fixes the smaller cyclic modules used throughout; they are never replaced by the full smaller tensor squares.

Proposition 2.1 (Source contracts). *The following hold for these specified modules.*

- (i) Every S^λ , $\lambda \vdash N_m$, occurs in W_m .
- (ii) For $s \in \{1, 2\}$ and $h = m - s \geq 1$, there is a surjection

$$W_m \twoheadrightarrow \text{Ind}_{S_{N_h} \times S_{N_m - N_h}}^{S_{N_m}} (W_h \boxtimes U_{m,s}). \quad (2.1)$$

Here $U_{m,s}$ is the cyclic module of the specified alternating row and column tensors on $B_m \setminus B_h$.

- (iii) $U_{m,1} \cong S^{(m)}$. For $m \geq 3$, $U_{m,2}$ contains at least one copy of every S^η with $\eta \vdash 2m - 1$ and $\ell(\eta) \leq 4$.
- (iv) If $K_{\alpha,\mu}$ is the Kostka number, then

$$[W_m : S^\lambda] \geq K_{\lambda^t, \rho_m}. \quad (2.2)$$

Input references and multiplicity justification. Part (i) is [1, dominance.tex, thm:cyclic-saxl]. Part (ii) is its band-quotient.tex, prop:band-quotient. That proposition gives the entire induced module, not an unspecified submodule: disjoint high-letter coordinate sectors realize the disjoint-coset model of induction. Part (iii) is path:one-band and prop:two-band in path-band.tex. Only this support inclusion is needed. The stronger identification of the whole width-two band using the multiplicity-free two-row square is inherited context [8, 9, 7], not an additional premise of our lower bound.

For (iv), recall the actual source dominance maps. If H_R is the row subgroup, its signed average gives $v_R \otimes u \in W_m$, where $u = \sum_{h \in H_R} h v_C \neq 0$ is H_R -fixed. The alternating line in $S^{\rho_m}|_{H_R}$ is one-dimensional. Reassociating a sign twist into the second factor therefore sends this tensor to a nonzero multiple of $z = v_R \otimes v_R$. Thus an isomorphic image of $W_m \otimes \text{sgn}$ contains $Z_m = \mathbb{C}[G_m]z$. The source's pair-letter/content projection sends z to

$$c_m t_m, \quad c_m = (-1)^{\sum_{r=1}^m \binom{r}{2}} \prod_{r=1}^m r! \neq 0,$$

where the orbit of t_m is the word basis of M^{ρ_m} . It follows that Z_m surjects onto M^{ρ_m} . Semisimplicity and Young's rule give (iv), with multiplicities. These maps appear in [1] under the labels

dom:signed-average, dom:repeated-row-cyclic,
dom:noncancellation, dom:permutation-quotient.

I21 proves additionally $Z_m \cong M^{\rho_m}$; we only need the surjection. $\square$

Write $c_{\nu,\eta}^\lambda$ for a Littlewood–Richardson coefficient. Following the multiplicity framework and max rule of I19, Section 4 [8], with the I21 Kostka refinement [9], define $R_1((1)) = 1$ and, for $m \geq 2$, define $R_m(\lambda)$ as the maximum of

$$\begin{aligned} & 1, \quad K_{\lambda^t, \rho_m}, \\ & \sum_{\nu \vdash N_{m-1}} R_{m-1}(\nu) c_{\nu, (m)}^\lambda, \\ & \sum_{\substack{\nu \vdash N_{m-2}, \eta \vdash 2m-1 \\ \ell(\eta) \leq 4}} R_{m-2}(\nu) c_{\nu, \eta}^\lambda \quad \text{if } m \geq 3. \end{aligned} \quad (2.3)$$

Induction and Proposition 2.1 give

$$R_m(\lambda) \leq [W_m : S^\lambda], \quad g(\rho_m, \rho_m, \lambda) \geq \max\{R_m(\lambda), R_m(\lambda^\dagger)\}. \quad (2.4)$$

Contributions within one quotient are added. Separate quotient bounds are maximized; their images have not been proved independent. The second inequality uses self-conjugacy of the staircase and sign invariance of the *ambient square*. No sign invariance of W_m is assumed.

For $h \geq 5$, let

$$\mathcal{E}_h = \{(N_h), (1^{N_h}), H_h, J_h\}, \quad H_h = (h, 1^{N_h-h}), \quad J_h = (h, 2, 1^{N_h-h-2}). \quad (2.5)$$

We prove the asymmetric invariant

$$R_h(\nu) \geq h - 1 \quad (h \geq 5, \nu \vdash N_h, \nu \notin \mathcal{E}_h). \quad (2.6)$$

The exact bases at $h = 5, 6$ and finite steps at $m = 7, 8$ appear in Section 7. Sections 3–6 prove the step for $m \geq 9$, using only (2.6) at $m - 1, m - 2$.

3 Wide diagrams

Suppose $m \geq 9$, $\lambda \vdash N_m$, $\lambda_1 > m$, and λ is not a row. If λ is not a rectangle, its horizontal m -strip predecessors are counted by

$$[x^m] \prod_i (1 + x + \cdots + x^{\lambda_i - \lambda_{i+1}}). \quad (3.1)$$

Indeed a horizontal strip is precisely the interlacing $\lambda_i \geq \nu_i \geq \lambda_{i+1}$. There are at least two positive gaps, and $0 < m < \lambda_1$. A product of at least two nonconstant initial geometric sums has every strictly interior coefficient at least two. For two factors this follows directly by counting bounded pairs with fixed sum; multiplying by another factor preserves the assertion by convolution. Thus there are at least two predecessors.

Put $h = m - 1$. At least one predecessor is outside $\mathcal{E}_h$. If (N_h) is a predecessor, λ has at most two rows and none of the other exceptions fits. If (1^{N_h}) is a predecessor, λ is a hook; its two predecessors have consecutive first-row lengths, which cannot both lie among the exceptional hook widths $1, h$. The remaining possibility that all predecessors are exceptional would require both H_h and J_h . Their interlacing implies $\lambda_2 \leq h$ and $\lambda_3 \leq 1$. But then

$$\nu' = (h + 1, 1^{N_h-h-1})$$

also interlaces λ : its first row lies between λ_1 and λ_2 , its second between λ_2 and λ_3 , and its remaining parts agree with J_h . It is not exceptional. One good predecessor contributes $m - 2$, and another contributes at least one. The one-band term proves $R_m(\lambda) \geq m - 1$.

If $\lambda = (a^b)$ is a rectangle, then $a \geq m + 1$ and $b \geq 2$. Choose

$$\begin{aligned} \eta_1 &= (m + 1, m - 2), & \nu_1 &= (a^{b-2}, a - m + 2, a - m - 1), \\ \eta_2 &= (m, m - 1), & \nu_2 &= (a^{b-2}, a - m + 1, a - m). \end{aligned}$$

Zero parts are omitted. The rectangle-complement LR identity gives $c_{\nu_i, \eta_i}^\lambda = 1$. Both predecessors are outside $\mathcal{E}_{m-2}$. For $b \geq 3$ their width is $a > m - 2$ and they are not rows. For $b = 2$ they have at most two rows, and ν_2 has two positive parts. The only potential one-row case for ν_1 would have $a = m + 1$ and $N_m = 2(m + 1)$, forcing $m = 4$. Thus the two-band term is at least $2(m - 3) \geq m - 1$. This finishes the wide case.

4 Two predecessors at high capacity

The next lemma incorporates the parent author-side construction [10], with its full proof. Let $C(\lambda) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

Lemma 4.1. *Let $m \geq 7$, $h = m - 2$, $n = N_h$, and $\lambda \vdash N_m$ satisfy $\lambda_1 \leq m$ and $C(\lambda) \geq 2m + 1$. There are two distinct $\nu \vdash n$, both outside $\mathcal{E}_h$, such that*

$$\lambda_i \geq \nu_i \geq \lambda_{i+4} \quad \text{for every } i. \quad (4.1)$$

Proof. Set $\mu = (\lambda_5, \lambda_6, \dots)$ and consider the skew diagram $S = \lambda/\mu$ with coordinate order: $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. Partitions $\mu \subseteq \nu \subseteq \lambda$ correspond exactly to order ideals of S . An interval between comparable cells lies inside the skew diagram, so this is also the order generated by row and column adjacencies. The desired rank is

$$k = n - |\mu| = C - (2m - 1), \quad k \geq 2, \quad C - k = 2m - 1 \geq 13.$$

Write $H_j = \lambda_j^\dagger$ and $L_j = \max(1, H_j - 3)$. Column j of S consists of rows $L_j, \dots, H_j$, so has at most four cells; both endpoint sequences are nonincreasing. Put $w = \lambda_1$.

We first prove that every rank r with $2 \leq r \leq C - 2$ has at least two ideals. Suppose A were the unique rank- r ideal, and write $B = S \setminus A$. Every $a \in A$ and $b \in B$ must satisfy $a \leq b$. The alternative $b < a$ contradicts downward closure. If they were incomparable, adjoining the relation $b < a$ would still give a partial order. A linear extension of that order has a first- r ideal of the original order different from A , a contradiction.

Let c be the greatest column meeting A and d the least column meeting B . Cross-comparability forces $c \leq d$. If $c < d$, column one lies wholly in A and column w wholly in B , whence $H_1 \leq L_w$. Since $H_1 \geq H_w \geq L_w$, this forces $H_1 = H_w = 1$ and $C = w \leq m$, impossible. The same argument applies if $1 < c = d < w$. If $c = d = 1$, all of A is a prefix of column one and column w lies in B , so $L_1 + r - 1 \leq L_w \leq L_1$, giving $r \leq 1$. If $c = d = w$, all of B is a suffix of column w and column one lies in A , so $H_w - |B| + 1 \geq H_1 \geq H_w$, giving $|B| \leq 1$. Finally $w = 1$ would give $C \leq 4$. Every case is impossible.

If no exceptional shape is an interval predecessor, this rank count at k finishes the proof. Suppose an exception fits. The row (n) cannot fit, since $n > m \geq \lambda_1$. If the column fits, then $\lambda_5 \leq 1$; if H_h fits, $\lambda_6 \leq 1$; if J_h fits, $\lambda_7 \leq 1$. Thus every column after the first has height at most six. Also

$$\ell(\lambda) \geq n - h = (m - 2)(m - 3)/2 \geq 10.$$

The four cells of S in column one consequently lie in rows at least seven. They are incomparable with all other cells of S , which lie in rows at most six and columns at least two. Therefore S is the disjoint union of a four-cell chain and a poset S' of size $C - 4$.

A finite poset has an ideal of every size from zero to its cardinality, by taking prefixes of a linear extension. Since $k \leq C - 4$, for each $a = 0, \dots, \min(4, k)$ combine a length- a prefix of the separated chain with a rank- $(k - a)$ ideal of S' . This gives at least $\min(4, k) + 1$ predecessors. At most two exceptions fit simultaneously: their lengths are $n, n - h + 1, n - h$, whereas every predecessor has length in the interval $[H_1 - 4, H_1]$, of width four, and $h \geq 5$. For $k \geq 3$, at least four predecessors therefore leave two usable ones.

It remains that $k = 2$, or $C = 2m + 1$. The construction gives three predecessors, whose lengths differ by at most two. The column cannot coexist with either H_h or J_h , because the corresponding length differences are $h - 1 \geq 4$ and $h \geq 5$. If both H_h, J_h fit, then $\lambda_6 \leq 1$ and $\lambda_5 \geq 1$. The number of cells of μ outside its first column is $\lambda_5 - 1$. Since J_h has h cells outside its first column and $|J_h/\mu| = 2$, we get $\lambda_5 \geq h - 1 = m - 3$. This contradicts

$$C \geq 4\lambda_5 \geq 4(m - 3) > 2m + 1 \quad (m \geq 7).$$

So at most one exception fits, leaving two usable predecessors. $\square$

Condition (4.1) says that every column of λ/ν has at most four cells. Filling each column, from top to bottom, by 1, 2, 3, 4 as needed gives a semistandard skew tableau. In row i , column j , the entry is $i - \nu_j^i$, which weakly increases with j . Hence $s_{\lambda/\nu}(1, 1, 1, 1) > 0$. By the LR expansion, some η with $\ell(\eta) \leq 4$ has $c_{\nu, \eta}^\lambda > 0$. The two predecessors of Lemma 4.1 contribute separately to the same two-band term in (2.3). Under the inductive hypothesis it is at least $2(m - 3) \geq m - 1$.

5 A uniform low-capacity Kostka bound

5.1 The configuration convention

For a partition β , put

$$Q_j(\beta) = \sum_i \min(j, \beta_i), \quad b_j(\beta) = \#\{i : \beta_i = j\}.$$

For partitions α, μ of the same integer, a configuration is a sequence of partitions $\nu^{(1)}, \nu^{(2)}, \dots$ with

$$|\nu^{(k)}| = \sum_{i>k} \alpha_i, \quad \nu^{(0)} = \mu, \quad (5.1)$$

and nonnegative vacancies

$$P_j^{(k)} = Q_j(\nu^{(k-1)}) - 2Q_j(\nu^{(k)}) + Q_j(\nu^{(k+1)}) \geq 0 \quad (k, j \geq 1). \quad (5.2)$$

There is *no upper cutoff on string lengths j at μ_1* . The size constraints alone make the relevant choices finite. The positive configuration formula is

$$K_{\alpha, \mu} = \sum_{\nu \text{ admissible}} \prod_{k, j \geq 1} \binom{P_j^{(k)} + b_j(\nu^{(k)})}{b_j(\nu^{(k)})}. \quad (5.3)$$

For precision about conventions, (5.3) follows from [4, Section 2.2, (2.4)–(2.6), Theorem 2.10] at $q = 1$. In their notation take target shape $\alpha^\dagger$ and rectangles $R_a = (1^{\mu_a})$, each a *column*. Their size constraints become (5.1); their $\xi^{(1)}(R)$ is μ and all other $\xi^{(k)}(R)$ are empty, giving (5.2). The multiplicity on their left is the coefficient of $s_{\alpha^\dagger}$ in $\prod_a e_{\mu_a}$. Applying the standard involution $\omega(s_\alpha) = s_{\alpha^\dagger}$, $\omega(h_r) = e_r$, and Young's rule identifies it with $K_{\alpha, \mu}$. KSS's partitions and vacancy indices range over all positive lengths. This explicitly supplies the unbounded convention needed below, independently of the matrix-width wording in [3, Section 1]. Kirillov's maximal configuration and one-cell perturbation [3, Corollary 1.3 and proof of Theorem 2.1] are the inherited operations we now quantify.

Lemma 5.1. *Let $d \geq 8$, $N = N_d$, and $L \vdash N$ satisfy $L_1 \leq d - 1$ and $L_1 + L_2 + L_3 + L_4 \leq 2d$. Then $K_{L^\dagger, \rho_d} \geq d$, except possibly at*

$$(1^N), \quad (2, 1^{N-2}), \quad (d-1, 1^{N-d+1}), \quad (d-1, d-1, 1^{N-2d+2}). \quad (5.4)$$

On the three noncolumn exceptions a single configuration has weight $d - 1$.

Proof. Write $w = L_1$, $t = \#\{i : L_i \geq 2\}$, $s = \#\{i : L_i = 1\}$, $\alpha = L^\dagger$, and

$$D_j = \sum_{i \leq j} ((\rho_d)_i - L_i).$$

For $4 \leq j \leq d-1$, monotonicity of prefix averages and the capacity bound give

$$D_j \geq \frac{j(d-j+1)}{2} \geq d-1. \quad (5.5)$$

For $j \geq d$, D_j is the area below row j of L , hence is nonnegative. If $w \leq d-2$, then $D_1 \geq 2, D_2 \geq 3, D_3 \geq 3$. If $w = d-1$, then $D_1 = 1, D_2 = d - L_2 \geq 1$ and $D_3 \geq d-2$: there are at least four rows and $L_1 + L_2 + L_3 \leq 2d-1$. In particular every D_j is nonnegative.

The maximal configuration is $\nu^{(k)} = (\alpha_{k+1}, \alpha_{k+2}, \dots)$. Its first vacancy is

$$P_j^{(1)} = D_j + \min(j, t+s) - \min(j, t). \quad (5.6)$$

It equals D_j for $j \leq t$ (and also for all j if $s = 0$). For $k \geq 2$ all vacancies are nonnegative, and the vacancy is zero at every occupied length of $\nu^{(k)}$. Its weight is consequently

$$M = \prod_{j=1}^t \binom{D_j + L_j - L_{j+1}}{L_j - L_{j+1}}. \quad (5.7)$$

If $s = 0$, the actual first-partition multiplicity at $j = t$ is $L_t - 1$, rather than L_t ; however $D_t = 0$, so both displayed and actual factors are one. Formula (5.7) remains valid. For $w = 1$ we have the column exception; henceforth $w \geq 2$.

The conjugate of $\nu^{(1)}$ is $q_j = \max(L_j - 1, 0)$. For $1 \leq u < v \leq t$, replace it by

$$q'_j = q_j + \mathbf{1}_{j=u} - \mathbf{1}_{j=v}. \quad (5.8)$$

If q' is a partition and $D_j \geq 2$ on $u \leq j < v$, this is an admissible distinct configuration. Indeed $Q_j(\nu^{(1)})$ increases by one exactly on $[u, v)$, so the first vacancies decrease by two and the second increase by one there. All other vacancies stay unchanged. That interval lies in $[1, t]$, where (5.6) equals D_j . Let W denote the weight of the modified configuration.

When $u = 1, v = t, t \geq 3$, q' is always a partition. The first-partition multiplicities at lengths 1 and $t-1$ are both at least one, at distinct positions. Their factors are at least $d-w-1$ and $D_{t-1}-1$, respectively. For $5 \leq t \leq d$, (5.5) gives

$$W \geq (d-w-1)(d-2). \quad (5.9)$$

For $t \geq d+1$, put $v_0 = L_t \geq 2$. Then $D_{t-1} = v_0 + s$. At every occupied second-partition length $j < t$, the old second vacancy was zero and becomes one. These factors multiply to at least

$$\begin{aligned} & \prod_{j < t} (1 + \max(L_j - 2, 0) - \max(L_{j+1} - 2, 0)) \\ & \geq 1 + \sum_{j < t} (\max(L_j - 2, 0) - \max(L_{j+1} - 2, 0)) = w - v_0 + 1. \end{aligned}$$

Thus, whenever this move is admissible,

$$W \geq (d-w-1)(v_0 + s - 1)(w - v_0 + 1) \geq (d-w-1)(w-1). \quad (5.10)$$

The last inequality uses $(v_0 - 1)(w - v_0 + 1) \geq w - 1$ for $2 \leq v_0 \leq w$.

Widths $3 \leq w \leq d-2$. If $t = 1$, (5.7) is $\binom{d-1}{w-1} \geq d$. Suppose $2 \leq t \leq 4$. There are unit rows because $N_d > 4(d-2)$ for $d \geq 8$. For $t = 2$, put $v = L_2 \geq 2$. The $j = 2$ factor is $\binom{2d-2-w}{v-1} \geq d$, since its upper argument is at least d and $1 \leq v-1 \leq d-3$. For $t = 3$ with $L_3 \geq 3$, we have $D_3 \geq d-2$ and final drop $L_3 - 1 \geq 2$, giving a factor at least $\binom{d}{2} \geq d$. If $L_3 = 2$, then $D_3 \geq 3d-5-2(d-2) = d-1$, again sufficient. For $t = 4$, $D_4 \geq 2d-6 \geq d$, and the final nonzero drop suffices.

Now let $t \geq 5$. All $D_j \geq 2$ for $1 \leq j < t$, by the first three bounds, (5.5), and the remaining row of length at least two when $j \geq d$. The move $u = 1, v = t$ is admissible. If $3 \leq w \leq d-3$, (5.9) gives $W \geq 2(d-2) \geq d$ when $t \leq d$. Otherwise (5.10) gives $W \geq 2(d-4) \geq d$, the minimum of $(d-w-1)(w-1)$ on this width range. If $w = d-2$, the capacity condition prevents the first four rows from all having length w . There is a positive drop at some $j \leq 3$, with $D_j \geq 2$, and hence $M \geq 3$. Equations (5.9) and (5.10) give respectively $W \geq d-2$ and $W \geq d-3$. The two distinct configurations add in (5.3), so $M + W \geq d$.

Width two. Write $L = (2^t, 1^s)$. The case $t = 1$ is a listed exception. For $2 \leq t \leq d-1$,

$$D_t = t(d-2) - t(t-1)/2 \geq d-1$$

by concavity and its endpoint values. For $t = d$, $D_t = s = d(d-3)/2 \geq d-1$. The final nonzero drop in (5.7) therefore gives at least d . For $t > d$ and $s \geq d-1$, the same factor is at least $s+1 \geq d$.

It remains that $t > d$ and $0 \leq s \leq d-2$. From $2t + s = N_d$ and $d \geq 8$ follows $t-2 \geq d$. Use instead

$$\nu^{(1)} = (t-2, 2), \quad \nu^{(k)} = \emptyset \quad (k \geq 2).$$

The size is correct. The first vacancies at $j = 1, 2$ are $d-4$ and $2d-9$. On $2 \leq j \leq t-2$ they equal the concave sequence $Q_j(\rho_d) - 2j - 4$, whose other endpoint is s ; for $j \geq t-2$ they equal s . They are all nonnegative. Second vacancies are $Q_j(\nu^{(1)}) \geq 0$. The distinct occupied lengths $2, t-2$ give weight

$$(2d-8)(s+1) \geq d.$$

This configuration illustrates why no cutoff at string length d may be imposed in (5.3).

Width $d-1$. For $t = 1$ we have the listed hook. If $t = 2$, put $v = L_2$. Then

$$M = (d-v) \binom{d-1}{v-1}.$$

It is at least $2(d-1) \geq d$ for $2 \leq v \leq d-2$; $v = d-1$ is the listed double-row exception. Suppose $t \geq 3$, put $v = L_2$ and $a = d-v$. As there are at least four rows, the capacity bound gives $L_3 + L_4 \leq a+1$, whence $L_3 \leq a$. Also $v \leq d-2$, so $a \geq 2$. If $a < d/2$, set $b = v - L_3 \geq d-2a \geq 1$. The first two maximal factors give

$$M \geq a \binom{a+b}{b} \geq a(a+b) \geq a(d-a) \geq d.$$

If $a \geq d/2$, use (5.8) with $u = 2, v = t$. It is a partition since the new second entry is $L_2 \leq d-2 = q_1$. The vacancy conditions follow from $D_2 = a \geq 2$, $D_3 \geq d-2$, (5.5), and a remaining row of length at least two. At length one the first multiplicity decreases by one while its vacancy stays one, giving the factor $a-1$. At length two the new first multiplicity is

at least one and the vacancy is $a - 2$, giving a factor at least $a - 1$. If $t = 3$, that multiplicity increases by two, and the same estimate holds. Hence $W \geq (a - 1)^2 \geq d$ for integral $a \geq d/2$, $d \geq 8$. Finally, direct substitution into (5.7) gives weight one on the column and $d - 1$ on each other shape in (5.4). All cases are covered. $\square$

6 Closing the uniform induction

Let $m \geq 9$ and assume (2.6) at $m - 1, m - 2$. The wide case was proved in Section 3. For $\lambda_1 \leq m$ and $C(\lambda) \geq 2m + 1$, Lemma 4.1 and its LR consequence prove the desired step. It remains that $\lambda_1 \leq m$ and $C(\lambda) \leq 2m$.

If $\lambda_1 \leq m - 1$, Lemma 5.1 gives $K_{\lambda^t, \rho_m} \geq m - 1$ except at the column, already in $\mathcal{E}_m$. If $\lambda_1 = m$, the first column of any semistandard tableau of shape λ^t and content ρ_m is forced to be $1, \dots, m$. Deleting it subtracts one from every content multiplicity and removes the last letter. Conversely any remaining tableau may have this column prepended: an entry in row i is at least i , so row weak increase is preserved. Thus, for $L = (\lambda_2, \lambda_3, \dots)$ and $h = m - 1$,

$$K_{\lambda^t, \rho_m} = K_{L^t, \rho_h}. \quad (6.1)$$

Here $|L| = N_h$ and $L_1 + L_2 + L_3 \leq h + 1$. If $L_1 = h + 1$, then $L_2 = L_3 = 0$, contradicting $|L| = N_h > h + 1$. If $L_1 = h$, then $L_2 + L_3 \leq 1$; this permits at most one further positive part, again contradicting the area. Therefore $L_1 \leq h - 1$. Also $L_4 \leq (h + 1)/3$, so the top-four sum is at most $4(h + 1)/3 \leq 2h$. Lemma 5.1 applies at $h \geq 8$.

Its column and width-two hook correspond exactly to H_m, J_m . The double- $(h - 1)$ exception cannot occur because its first three parts sum to $2h - 1 > h + 1$. The remaining width- $(h - 1)$ hook gives

$$\lambda = (m, m - 2, 1^{N_m - 2m + 2}). \quad (6.2)$$

This family is handled inside one two-band quotient, as follows.

Lemma 6.1. *For $m \geq 5$, set $h = m - 2$, $n = N_h$, and $\nu = (2, 1^{n-2})$. For λ in (6.2),*

$$\sum_{\substack{\eta \vdash 2m-1 \\ \ell(\eta) \leq 4}} c_{\nu, \eta}^\lambda \geq 2.$$

Proof. Since $\lambda = (m, m - 2, 1^{n+1})$, the skew shape λ/ν is the disjoint union of a bottom column of four cells and $D = (m, m - 2)/(2, 1)$. Jacobi–Trudi and Pieri give

$$\begin{aligned} s_D &= h_{m-2}h_{m-3} - h_m h_{m-5} \\ &= s_{(m-1, m-4)} + s_{(m-2, m-3)}. \end{aligned}$$

Multiplying by e_4 gives, by adding a vertical four-strip, the two distinct constituents $(m, m - 3, 1, 1)$ and $(m - 1, m - 2, 1, 1)$, each with coefficient at least one. $\square$

For $m \geq 7$, ν is outside $\mathcal{E}_{m-2}$. In the uniform induction its contribution is at least $2(m - 3) \geq m - 1$. This completes the step at every $m \geq 9$.

7 Exact finite starts and reproducibility

This section states the finite part of the proof explicitly. It is a finite exhaustive calculation, not evidence substituted for a universal argument. The supplied standard-library Python code uses integers and `Fraction` throughout; it requires no downloaded data, private source program, network access, floating point, or modular arithmetic.

7.1 Algorithms and their mathematical contracts

Partitions of n are enumerated by selecting their first part j and recursively enumerating partitions of $n - j$ with largest part at most j . Horizontal-strip predecessors are enumerated by $\lambda_i \geq \nu_i \geq \lambda_{i+1}$ with prescribed area. Deleting the cells occupied by the largest entry of a semistandard tableau therefore gives the exact Kostka recurrence

$$K_{\lambda, (\mu_1, \dots, \mu_r)} = \sum_{\lambda/\nu \text{ horizontal } \mu_r\text{-strip}} K_{\nu, (\mu_1, \dots, \mu_{r-1})}.$$

The four-strip predecessor enumeration imposes weak decrease, total area, and $\lambda_i \geq \nu_i \geq \lambda_{i+4}$ directly.

Characters are computed by Murnaghan–Nakayama. For a partition with l parts, form the beta set $\{\lambda_i + l - i : 1 \leq i \leq l\}$. Replacing a bead x by an unoccupied nonnegative $x - k$ enumerates a removable k -border strip, with sign $(-1)^b$ where b is the number of crossed beads. Recursion gives $\chi^\lambda(\mu)$. For cycle type $\mu = (1^{a_1} 2^{a_2} \dots)$ put $z_\mu = \prod_j j^{a_j} a_j!$. Then

$$g(\rho, \rho, \lambda) = \sum_{\mu \vdash n} \frac{\chi^\rho(\mu)^2 \chi^\lambda(\mu)}{z_\mu}. \quad (7.1)$$

For class functions a, b on symmetric groups of two prescribed degrees,

$$\chi_{\text{Ind}(a \boxtimes b)}(\mu) = \sum_{u \sqcup v = \mu} \frac{z_\mu}{z_u z_v} \chi_a(u) \chi_b(v). \quad (7.2)$$

The sum distributes the multiset of cycles between the two degrees. Character inner products then compute the exact induced multiplicities in (2.3). At every induction the verifier checks integrality, nonnegativity, and the induced dimension. Small-square calculations also check staircase orthogonality, the hook-length dimension, the total square dimension, and both linear coefficients.

7.2 Finite conclusions

Formula (7.1) gives the minima 1, 2, 3 at $m = 2, 3, 4$. Starting from R_1 , the verifier computes the four terms of (2.3) exactly through $m = 6$. It proves (2.6) at $h = 5, 6$ for all 172 and 788 nonexceptional labels.

At $m = 7, 8$, each nonexceptional label is checked in the following order:

1. the one-band sum, giving weight $m - 2$ to each nonexceptional predecessor and weight one to each exception;
2. the exact Kostka number;
3. the four-strip predecessor sum, giving weight $m - 3$ to each nonexceptional predecessor and weight one to each exception;
4. the explicit critical-family identity of Lemma 6.1.

The third test is a valid lower bound on one two-band quotient: each predecessor has at least one four-row LR constituent, as proved after Lemma 4.1. The fourth test invokes a proved symbolic identity; it is not inferred from the existence of a shape in a list. The exhaustive counts are

m	N_m	all labels	one-band	Kostka	four-strip	critical
7	28	3718	2349	1120	244	1
8	36	17977	12161	4948	863	1

The four excluded labels are exactly $\mathcal{E}_m$ in each row. Every retained label has certified lower bound at least $m-1$. This establishes the two finite induction steps without ever substituting ambient-square multiplicities for the smaller cyclic multiplicities.

7.3 Reproduction and limits

The public supplement contains the verifier, its three computational modules, and the complete certificate:

```
verify-finite-proof.py, multiplicity-diagnostic.py,  
quotient-diagnostic.py, exception-induction-diagnostic.py,  
finite-proof-certificate.json.
```

From the supplement root, run

```
python3 -B code/verify-finite-proof.py
```

The default mode recomputes all finite claims and compares every output entry with the certificate; it does not merely trust a stored pass flag. The certificate SHA256 is

```
5ffb18808a16add627497b29c82ee389  
187ad8f4d6df0f6fa52cc6ab49d99263
```

A separate supplied character crosscheck compares grid-cell border strips and beta-set characters through degree ten. The independent proof review also recomputed the starts using permutation characters with triangular Kostka inversion and direct LR tableaux; its reported critical-family weighted two-band values are 11 and 13 at $m = 7, 8$. These provide separate checks, not additional hypotheses and not a substitute for the finite algorithms or the uniform proof above.

No claim is made that the entire theorem is formalized. The separately anchored Lean support [12] contains 17 named declarations (two definitions and fifteen theorems). Its mathematical coverage is generic finite-poset rank interpolation, the unique-rank ordered-cut implication, and the two-ideal reduction. The geometric no-cut condition for our skew diagrams is not formalized. Neither are exception exclusion, the configuration formula, representation theory, source maps, finite character calculations, or the complete induction. The scoped independent review audited compiler-generated constants as well as the named declarations; this remains support for the combinatorial proof, not a formalization of Theorem 1.1.

8 Conclusion and attribution

The finite starts and Sections 3–6 prove (2.6) for every $h \geq 5$ by strong induction. For $m \geq 5$, a label other than row and column is either outside $\mathcal{E}_m$ or is H_m or J_m . In the latter cases its transpose has first row $N_m - m + 1$ or $N_m - m$, respectively, greater than m , and is not a row. It is therefore outside $\mathcal{E}_m$. Equation (2.4) proves (1.1). The cases $m = 2, 3, 4$ are supplied by Section 7.

For sharpness, $S^{(N-1,1)}$ is the nontrivial summand of the natural permutation module. Branching and Frobenius reciprocity give

$$g(\rho_m, \rho_m, (N_m - 1, 1)) = \dim \text{End}_{S_{N_m-1}}(\text{Res } S^{\rho_m}) - 1 = m - 1,$$

because the staircase has m distinct removable corners and branching is multiplicity-free. This finishes Theorem 1.1.

Inherited operations and the residual estimate. The cyclic support, specified full induced quotients, and pair-letter projection are source results [1]. I19’s multiplicity recursion and max rule [8], I21’s exact dominance module [9], and Kirillov’s maximal configuration and one-cell perturbation [3] are explicitly inherited. The companion [2] already uses capacity and balance to prove support. The residual here is the uniform four-exception estimate for the cyclic recursion, including the predecessor count and the low-capacity quantitative bound. The high-capacity lemma is credited parent author assistance [10]; the remaining configuration analysis and full induction originated in the author packet [11]. Neither parent assistance nor finite replay is an independent review.

Literature comparison and claim boundary. Vallejo’s fixed-tail formulas and asymptotics [5, Section 8] allow a fixed partition tail while the staircase order grows; they do not supply the all-label quantifier used here. Bessenrodt–Bowman–Sutton [6, Section 5] give lower bounds for specified families, and Bessenrodt–Bowman [7, Theorem 7.11] gives the stronger two-row square splitting used in the earlier project analysis. The corner value $m - 1$ is classical and already explicit in this literature. The independent contribution comparison found the full uniform estimate new to this project, after deducting those operations. Neither that bounded comparison nor this article demonstrates new-to-literature priority. No campaign acceptance or publication decision is asserted here.

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