

Two Seymour vertices in every sink-free oriented graph

An exceptional-boundary and aggregate-fiber argument

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Abstract

We prove, relative to two explicitly stated results of the pinned OpenAI manuscript on Seymour’s conjecture, that every nonempty finite sink-free oriented graph has two distinct vertices whose exact second out-neighborhood is at least as large as their first. This is Conjecture 2 of Dara, Francis, Jacob and Narayanan. The inputs are the ordinary one-vertex Seymour theorem and a finite-relation pruning lemma; their truth is stipulated here. A minimum counterexample has a unique good vertex with one successor. Deleting that vertex leaves weak subset deficits together with a strict inequality for subsets that do not reach the successor in one or two steps. In a transitive blowup, compatibility with the diagonal selects many fibers with this strict inequality. Their total gain dominates the possible losses of all remaining fibers, allowing the established pruning objective to close. All deductions from the two inputs are proved below. Finite diagnostics and formal supporting lemmas are distinguished from the full graph theorem.

1 Statement, notation and dependencies

An *oriented graph* is a finite directed graph without loops, parallel arcs or opposite pairs of arcs. For a vertex v of D , let $N_{1,D}(v)$ be its set of out-neighbors and let $N_{2,D}(v)$ consist of the vertices at *directed distance exactly two* from v . In particular, a first neighbor is not counted again in N_2 . A vertex is *good* if $|N_{2,D}(v)| \geq |N_{1,D}(v)|$. A sink has no outgoing arc.

Theorem 1. *Assume Premises 1 and 2. Every nonempty finite oriented graph with no sink has at least two distinct good vertices.*

There is no strong-connectivity, degree, missing-edge or size restriction in this conclusion. Sink-freeness is essential: a transitive tournament has only one good vertex, its sink. The statement is the already formulated Conjecture 2 of Dara et al. [2], not a new conjecture.

For any such graph K and vertex subset S , define the outgoing image and its iterates by

$$F_K(S) = \bigcup_{v \in S} N_{1,K}(v), \quad F_K^2(S) = F_K(F_K(S)).$$

We do not remove S from either image. Put

$$\delta_K(S) = |F_K^2(S) \setminus F_K(S)| - |F_K(S) \setminus S|. \quad (1)$$

For a singleton this is $|N_{2,K}(v)| - |N_{1,K}(v)|$: a length-two return to v would require opposite arcs, and loops are absent. For larger sets, the image notation in (1) is the definition; no distance interpretation for a set is needed.

Our only imported mathematical inputs are the following two statements from OpenAI [1], at the immutable revision recorded in Appendix A. They are stipulated true for this deduction; their proofs are not part of this paper’s claim of independent verification.

Premise 1 (Ordinary Seymour theorem). *Every nonempty finite oriented graph has at least one good vertex.*

This is `thm:main` in the source's `00-introduction.tex`. We next give the full pruning input in notation convenient for its later use.

Premise 2 (Finite-relation pruning). *Let X be any finite set equipped with a binary relation $\rightarrow$; loops or opposite pairs are allowed for this premise. Let $R, C \subseteq X \times X$ be two separately counted families. A left member $(p, j) \in R$ and a right member $(i, s) \in C$ conflict precisely when $p \rightarrow i$ and $s \rightarrow j$. Let*

$$\begin{aligned} Z &= \{(i, j) : \exists (p, j) \in R, (i, s) \in C, p \rightarrow i, s \rightarrow j\}, \\ B &= \{(p, s) : \exists (p, j) \in R, (i, s) \in C, p \rightarrow i, s \rightarrow j\}. \end{aligned}$$

A left member (p, j) covers (i, j) if $p \rightarrow i$, and a right member (i, s) covers (i, j) if $s \rightarrow j$; these are the only covering incidences. There exist subfamilies $R' \subseteq R$ and $C' \subseteq C$ with no remaining conflict, retaining every member that had no conflict initially, and satisfying

$$|R \setminus R'| + |C \setminus C'| \leq |B| + |\{u \in Z : u \text{ is not covered by } R' \text{ or } C'\}|. \quad (2)$$

A member appearing on both sides is counted separately on each side.

This is `lem:pruning` in `02-pruning.tex`; its initial charge set is called H there and B here to avoid confusion with a graph below. The premise applies to every finite binary relation, with no additional orientation, connectivity or nonemptiness hypothesis.

2 A minimum counterexample and protected subsets

Suppose the conclusion of Theorem 1 is false. Choose a sink-free counterexample D minimizing its number of vertices and, subject to that, its number of arcs. Premise 1 implies that D has exactly one good vertex, denoted by z .

The graph D is strongly connected. Otherwise, a terminal strongly connected component C is proper. It cannot be a singleton, because D is sink-free; its induced graph is sink-free. Both out-neighborhoods of each vertex in C stay inside C , so minimum order gives two good vertices of D , a contradiction.

Every vertex $x \neq z$ has outdegree at least two. Indeed, outdegree zero is excluded. If $N_{1,D}(x) = \{y\}$, then $N_{2,D}(x) = N_{1,D}(y)$: loops and opposite arcs exclude both x and y from the set on the right. It is nonempty, so x would be good. On the other hand, z has outdegree exactly one. If it had at least two outgoing arcs, deleting one would leave a sink-free graph with fewer arcs. At any other vertex the first neighborhood would be unchanged and the second could not enlarge. All vertices other than z would remain bad, contradicting arc minimality. Write $z \rightarrow w$ for its unique outgoing arc.

Lemma 2 (Protected strict deficit). *For every nonempty $S \subseteq V(D) \setminus \{z\}$, one has $\delta_D(S) < 0$.*

Proof. Fix S , and write $I = F_D(S) \cap S$ and $E = F_D(S) \setminus S$. Strong connectivity and the fact that S is proper imply $E \neq \emptyset$. For $T \subseteq E$ put $g(T) = |F_D(T) \setminus (I \cup T)|$.

For each proper $T \subsetneq E$, delete all arcs from S to $E \setminus T$, and call the resulting graph D_T . At least one arc is removed. If D_T has a sink, this sink lies in S and is newly good. If D_T is sink-free, minimality gives at least two good vertices in it. Outside S , first neighborhoods are unchanged and

second neighborhoods cannot improve, so only z can be good there. In either case there is a newly good vertex $v \in S$. Set

$$Q = N_{1,D}(v) \cap (E \setminus T).$$

This is precisely the set of first neighbors lost by v . It is nonempty, since otherwise v could not improve. Put $d_i = |N_{i,D}(v)|$. Since $v \neq z$, $d_2 \leq d_1 - 1$.

The surviving first neighbors of v lie in $I \cup T$. A two-step path through $I \subseteq S$ ends in $I \cup T$, and one through T ends in $F_D(T)$. Thus all surviving two-step endpoints lie in $I \cup T \cup F_D(T)$. A newly counted exact second neighbor must be a lost first neighbor, and therefore lies in $Q \cap F_D(T)$. Conversely, every member of

$$L = F_D(Q) \setminus (I \cup T \cup Q \cup F_D(T))$$

was formerly an exact second neighbor and is no longer reachable in two steps. Such a member cannot equal v , since that would give opposite arcs between v and a member of Q . Goodness in D_T consequently gives

$$d_1 - |Q| \leq |N_{2,D_T}(v)| \leq d_2 - |L| + |Q \cap F_D(T)|. \quad (3)$$

On adjoining Q to T , exactly $Q \cap F_D(T)$ is removed from the set counted by $g(T)$, and exactly L is added. Hence

$$g(T \cup Q) - g(T) = |L| - |Q \cap F_D(T)| \leq |Q| - 1.$$

Starting at $T = \emptyset$, repeat this construction, always forming D_T afresh from D . Each nonempty Q is disjoint from the current T , so finitely many steps reach $T = E$. There is at least one step, and $g(\emptyset) = 0$; thus $g(E) \leq |E| - 1$. Finally $I \subseteq S$ implies $F_D(I) \subseteq F_D(S)$, whence

$$F_D^2(S) \setminus F_D(S) = F_D(E) \setminus (I \cup E).$$

Its cardinality is $g(E)$, proving the lemma. $\square$

This argument is an adaptation of the source's `lem:subset-deficit`, with the unique exceptional vertex protected. It does not assert a strict deficit for sets containing z .

3 Retaining the exceptional boundary

Let $H = D - z$, and write $F = F_H$. Every remaining vertex had at least two outgoing arcs in D and loses at most one, so H is sink-free. For a nonempty $S \subseteq V(H)$ write $A = F(S)$, $A_2 = F^2(S)$ and set

$$a = \mathbf{1}_{\{z \in F_D(S)\}}, \quad q = \mathbf{1}_{\{a=1, w \notin A \cup A_2\}}, \quad b = \mathbf{1}_{\{a=0, z \in F_D(A)\}}.$$

The first image in D is A with z adjoined exactly when $a = 1$. The H -part of the second image is A_2 with w adjoined exactly when $a = 1$. Its possible z -part is removed by the first image when $a = 1$. These observations give the exact identities

$$|F_D^2(S) \setminus F_D(S)| = |A_2 \setminus A| + q + b, \quad (4)$$

$$|F_D(S) \setminus S| = |A \setminus S| + a. \quad (5)$$

Lemma 2, including integrality of cardinalities, now yields

$$\delta_H(S) \leq -1 + a - q - b \leq 0. \quad (6)$$

There is extra information beyond this weak bound:

$$w \notin F(S) \cup F^2(S) \implies \delta_H(S) \leq -1. \quad (7)$$

Indeed, if $a = 1$, then $q = 1$; if $a = 0$, the first bound in (6) is already at most -1 .

The graph H is strongly connected as well. Choose a terminal strongly connected component C in it. Sink-freeness makes C nontrivial, and every vertex in it has an internal predecessor. Hence $F(C) = C = F^2(C)$. Strong connectivity of D forces an arc from C to z , since there is no other possible outgoing destination outside C . Thus $a = 1$ for $S = C$. As $\delta_H(C) = 0$, (6) forces $q = 0$, so $w \in C$. Then $C \cup \{z\}$ is outgoing-closed in D , which by strong connectivity implies $C \cup \{z\} = V(D)$. Consequently $C = V(H)$. In particular, every vertex of H has positive indegree. Write $n = |V(H)|$.

4 A transitive blowup and its fiber estimates

Set $m = n^2 + 2$. Replace each vertex p of H by a transitive tournament on labels $1, \dots, m$, with arcs from smaller to larger labels. For every base arc $p \rightarrow q$, put all arcs from the p -block to the q -block. Denote this oriented graph by G , its vertex set by X , and its outgoing image operator by J . Thus $|X| = nm$, and all vertices of G have positive indegree. For $U \subseteq X$ define

$$c(U) = |U| + |J^2(U)| - 2|J(U)|.$$

Lemma 3. *Let U be a nonempty proper subset of X , and let $S = \{p \in V(H) : U \text{ meets the } p\text{-block}\}$. If $S \neq V(H)$, then*

$$c(U) \leq m\delta_H(S) + |S \setminus F(S)| \leq m\delta_H(S) + n. \quad (8)$$

If $S = V(H)$, then $c(U) = |U| - |X| < 0$. In all cases $c(U) \leq n$.

Proof. Let T_p be the labels selected by U in block p , and let f be the internal image operator of the transitive tournament. If a nonempty label set T has least element t , then $f(T) = \{t+1, \dots, m\}$. For $t < m$, $f^2(T) = \{t+2, \dots, m\}$ and

$$|T| - 2|f(T)| + |f^2(T)| = |T| - (m - t) - 1 \leq 0.$$

For $t = m$, the expression is 1. In particular it is always at most 1.

Put $W = F(S) \cup F^2(S)$. The first image fills every $F(S)$ -block and has label set $f(T_p)$ in a block $p \in S \setminus F(S)$. A two-step path with two external arcs ends in $F^2(S)$, one with exactly one external arc ends in $F(S)$, and a path wholly internal contributes $f^2(T_p)$. Allowing complete W -blocks therefore gives an upper bound:

$$\begin{aligned} c(U) &\leq m(|W| - 2|F(S)|) + \sum_{p \in S} |T_p| \\ &\quad - 2 \sum_{p \in S \setminus F(S)} |f(T_p)| + \sum_{p \in S \setminus W} |f^2(T_p)|. \end{aligned}$$

Extend the last sum to $S \setminus F(S)$; this only increases it. Bound $|T_p| \leq m$ on $S \cap F(S)$ and apply the internal inequality to all remaining blocks. The result is

$$c(U) \leq m(|F^2(S) \setminus F(S)| - |F(S) \setminus S|) + |S \setminus F(S)|,$$

which proves (8). If $S = V(H)$, positive indegrees in H give $F(S) = V(H)$, hence $J(U) = X$ and $J^2(U) = X$. This proves the full-support assertion. Finally use (6) for the bound $c(U) \leq n$. $\square$

5 Strictness after summing compatible fibers

For a family $P \subseteq X \times X$, define its two coordinate images by

$$\begin{aligned} J_1(P) &= \{(i, j) : \exists (p, j) \in P, p \rightarrow i\}, \\ J_2(P) &= \{(i, j) : \exists (i, s) \in P, s \rightarrow j\}. \end{aligned}$$

Call P, Q *compatible* when $J_1(P) \cap J_2(Q) = \emptyset$. The diagonal $\Delta = \{(x, x) : x \in X\}$ is compatible with itself because G has no opposite arcs. In this section let P, Q be compatible and contain Δ .

For $j, i \in X$, respectively write $P_j = \{p : (p, j) \in P\}$ and $Q_i = \{s : (i, s) \in Q\}$. Each column P_j and row Q_i is nonempty and proper. The diagonal gives nonemptiness. If $P_j = X$, positive indegrees imply $X \times \{j\} \subseteq J_1(P)$. An in-neighbor $t \rightarrow j$ gives $(t, j) \in J_2(\Delta)$, a conflict. If $Q_i = X$, then $\{i\} \times X \subseteq J_2(Q)$, and an in-neighbor $t \rightarrow i$ gives $(i, t) \in J_1(\Delta)$, again a conflict.

Fix $j = (w, a)$ with $2 \leq a \leq m$. No member $x \in P_j$ can have a two-step walk $x \rightarrow t \rightarrow j$: the left member (x, j) and right diagonal member (t, t) would conflict. Consequently the support S of P_j satisfies

$$w \notin F(S) \cup F^2(S). \quad (9)$$

For the first image, a base arc $p \rightarrow w$ from a supported block lifts from any chosen p -vertex through $(w, a - 1)$ to (w, a) . For the second image, a base walk $p \rightarrow q \rightarrow w$ lifts through any vertex in the q -block. Either would contradict the exclusion just proved. The support is nonempty, and it is proper since it omits every base in-neighbor of w , of which there is at least one. Equations (7) and (8) therefore give

$$c(P_{(w,a)}) \leq -m + n \quad (2 \leq a \leq m).$$

For the right row anchored at $i = (w, a)$, a walk $s \rightarrow t \rightarrow i$ would make the right member (i, s) conflict with the left diagonal member (t, t) : both coordinate images contain (i, t) . The same lifts then prove (9) for the row support and give $c(Q_{(w,a)}) \leq -m + n$.

There are nm fibers on each side. Lemma 3 bounds each by n , while the $m - 1$ specified anchors improve this bound by m . Thus

$$\sum_{j \in X} c(P_j) \leq n^2 m - m(m - 1) = -m < 0, \quad (10)$$

$$\sum_{i \in X} c(Q_i) \leq n^2 m - m(m - 1) = -m < 0. \quad (11)$$

Coordinate images act separately on the relevant fibers, so these imply

$$|P| + |J_1^2(P)| < 2|J_1(P)|, \quad |Q| + |J_2^2(Q)| < 2|J_2(Q)|. \quad (12)$$

This is a total-family conclusion. It requires no assertion that every individual nonempty proper subset of G has negative c .

6 The extremal pruning contradiction

There are finitely many compatible pairs containing Δ , and at least one exists. Choose such a pair P, Q maximizing $M + d$, where

$$M = |P| + |Q|, \quad d = |X|^2 - |J_1(P)| - |J_2(Q)|.$$

Here and below $|X|^2$ means $(|X|)^2 = |X \times X|$. Compatibility makes d exactly the number of points uncovered by the two coordinate images. Form the complementary families

$$R = (X \times X) \setminus J_2^2(Q), \quad C = (X \times X) \setminus J_1^2(P).$$

Adding (12) yields

$$|R| + |C| > M + 2d. \quad (13)$$

Both complements contain Δ . For example, if $(v, v) \in J_2^2(Q)$, then some $(v, s) \in J_2(Q)$ has $s \rightarrow v$. But $(s, s) \in P$ then puts (v, s) in $J_1(P)$, a conflict. For the other complement, if $(v, v) \in J_1^2(P)$, then some $(s, v) \in J_1(P)$ has $s \rightarrow v$, while $(s, s) \in Q$ puts (s, v) in $J_2(Q)$.

Moreover, every diagonal member in these new families is conflict-free. A left diagonal member $(v, v) \in R$ conflicting with $(i, s) \in C$ would give $s \rightarrow v \rightarrow i$. Since $(s, s) \in P$, this puts $(i, s) \in J_1^2(P)$, contrary to its membership in C . A right diagonal member $(v, v) \in C$ conflicting with $(p, j) \in R$ gives $p \rightarrow v \rightarrow j$; from $(p, p) \in Q$ we obtain $(p, j) \in J_2^2(Q)$, again a contradiction.

Apply Premise 2 to R, C . For any conflict $(p, j) \in R, (i, s) \in C$, where $p \rightarrow i$ and $s \rightarrow j$, its initial charge (p, s) lies outside $J_1(P) \cup J_2(Q)$. Indeed, membership in $J_1(P)$ would give $(u, s) \in P$ with $u \rightarrow p$, hence $(i, s) \in J_1^2(P)$. Membership in $J_2(Q)$ would give $(p, t) \in Q$ with $t \rightarrow s$, hence $(p, j) \in J_2^2(Q)$. Both are impossible. Thus the initial charge set B in the premise has size at most d .

The premise gives compatible retained families $P' \subseteq R$ and $Q' \subseteq C$, retaining the entire diagonal. Put

$$d' = |X|^2 - |J_1(P')| - |J_2(Q')|.$$

Every point of the premise's set Z left uncovered by the retained members is outside $J_1(P') \cup J_2(Q')$. Their number is therefore at most d' . Consequently at most $d + d'$ members were deleted, with the two sides counted separately. Combining this with (13) gives

$$|P'| + |Q'| + d' \geq |R| + |C| - (d + d') + d' > M + d.$$

The retained pair is admissible for the same maximum, a contradiction. This rules out the assumed counterexample and proves Theorem 1. $\square$

7 Attribution and the scope of the contribution

Dara et al. [2] state exactly our conclusion as Conjecture 2 and its unique-good/sink version as Conjecture 3. Their Lemma 18 describes the singleton boundary after deleting a vertex: a good vertex that becomes bad on reinsertion must point to the deleted vertex and see its successors in its first two neighborhoods. Their Theorem 6 proves a deletion-survival statement when the deleted graph is a tournament missing a matching, using safe completions, modules and median-order arguments; Corollary 5 supplies two good vertices in that restricted class. Their Proposition 8 uses minimality and arc deletion in comparing the conjectures. These are antecedents, not new constructions claimed here.

Seacrest's subset-deletion method [3, Lemma 4] is an antecedent of the protected-subset argument. Brantner et al. [4, Theorems 4.1–4.3] give minimum-counterexample properties and lexicographic amplification. OpenAI [1] develops the strict subset-deficit reduction, transitive blowup estimate, compatible-pair objective, complement step and pruning closure used here. The source's route begins with strict deficits for every relevant subset of an all-bad counterexample and makes each individual blown-up subset strictly concave.

The added operation here is the conjunction of (4)–(7) with (9)–(11). A deleted exceptional vertex leaves only weak deficits globally, but also a distinguished successor. Compatibility selects $m - 1$ anchor fibers in its block; their combined gain $m(m - 1)$ dominates the residual n^2m . Strictness is obtained after summing the constrained fibers. Neither standard minimality nor the final imported pruning operation receives separate novelty credit.

For additional comparison, Guo, Kang and Zwaneveld [5] study pointwise tightness under products and module replacements; these are product precedents rather than the exceptional-anchor selection above. Tahara [6] studies excess margins and restricted constructions. An excess bound and a bound on the number of good vertices are different: one good vertex can contribute several units of excess. The literature comparison covers the named primary bodies and is bounded. It is not a certification of worldwide priority. The theorem is presented relative to its stipulated inputs, not as a wholly independent proof of ordinary SNC.

8 Validation and formalization boundary

The universal proof is Sections 2–6. Computation is not used to infer the theorem. Two independently written finite diagnostics accompany it. The author’s bitmask checker verifies 352,224 deletion-boundary identities across 23,664 small orientations and verifies a six-vertex counterexample to a discarded shortcut. An independent reviewer’s set-based checker verifies 15,129 deletion endpoint and increment cases, 198,936 blowup subset cases, and 151,362 anchor exclusion cases. The latter enumerates orientations on three and four base vertices, with internal block sizes three and two. These finite tests detect local errors of signs, endpoints or directions; they do not verify the unrestricted assertion by enumeration.

For clarity, the discarded shortcut asserted that strong connectivity plus $\delta_H(S) \leq 0$ for all nonempty S forces zero singleton deficits. It is false: with vertices $0, \dots, 5$ and outgoing bitmasks $(36, 25, 10, 17, 37, 10)$, all 63 nonempty subset deficits are nonpositive but the singleton deficits are $(0, -1, 0, 0, -1, 0)$. The proof above uses the stronger exceptional-successor information (7), not this shortcut. The counterexample is not an S2 counterexample.

Formal support and its limits. The accompanying Lean 4.34.1 project contains eleven supporting theorems and seven control theorems, with mathlib pinned to revision

d13f23b723b8a846827a245b89c10fc7d3f11612.

The implementation’s fresh-source build and type/axiom receipts report only `propext`, `Classical.choice` and `Quot.sound` as kernel axioms. No upstream theorem or new conclusion is installed as an axiom. These are implementation checks; independent formal audit is separate.

The three boundary theorems prove the integer consequences of the assumed identity $\delta_D = \delta_H + q + b - a$, including the complete indicator case split. Three aggregate theorems treat arbitrary finite index sets, a distinguished subset of at least $m - 1$ improved fibers, nm fibers in total, and the bound $\sum c \leq -m < 0$ for $m = n^2 + 2$. Two fiber theorems convert negative total finite-fiber defects into strict sums and tagged-family cardinalities. Three closure theorems prove complement cardinality gain and objective improvement from explicitly assumed disjointness, strict image inequalities, retained subsets, deletion budget and maximality.

The full graph theorem is *not formalized*. The graph-to-boundary identity, minimum-counterexample argument, protected subset deficit, strong connectivity after deletion, transitive-blowup image estimate, compatibility and anchor exclusion, coordinate-image interpretation, pruning existence, diagonal retention, charge-set bounds and admissibility remain in the paper

proof. The control theorems expose false weakenings; they are not graph counterexamples or extra frontier results. The package’s `formal/coverage.json` lists every declaration and its exact assumptions.

The discovery, internal adversarial proof review, literature comparison and manuscript preparation used AI assistance. Internal review verdicts are workflow checks, not external peer review or journal acceptance. The local package records exact candidate and review identities and distinguishes inherited checkers from the portable packaging adapter. No public upload or submission is asserted.

A Source identity and reproducibility

The fixed source is `openai/math`, revision

```
fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb.
```

Its manuscript directory is

```
preprints/A-proof-of-Seymours-second-neighborhood-conjecture-September-23-2026.
```

Under `build/source/`, the two imported statements are `00-introduction.tex`, `thm:main`, and `02-pruning.tex`, `lem:pruning`. The operation antecedents are `01-reduction.tex`, notably `lem:subset-deficit`, and `03-incompatibility.tex`. Weighted and directed-cycle consequence sections were checked for scope overlap; neither is an extra premise here. The complete family inventory identifies no separate companion manuscript. The package records source URLs, Git blob identities and SHA-256 values; third-party source files and reading PDFs are not redistributed in it.

From a clean extraction of the public package, with Python 3.10 or later, run

```
python3 -B support/run_checks.py
```

This authenticates the preserved checker copies and adapter, runs both finite diagnostics, and compares their structured outputs with the fixed expected counts. It fails if a required file, check or expected result is absent. The separate integrity command

```
python3 -B support/verify_package.py
```

checks exact payload membership and hashes. Neither command requires private reading copies or the original workspace. Integrity is not mathematical proof or a certificate of origin; a recipient should compare the detached manifest hash with a trusted record before executing code. Build and rendering instructions, exact coverage, rights and local submission metadata are in the accompanying README.

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References

- [1] OpenAI, *A proof of Seymour's second-neighborhood conjecture*, September 23, 2026. OpenAI Math Release, revision fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb. [Pinned manuscript and sources](#).
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