

The crossing number of the complete graph minus an edge

Myeongjin Kang

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Abstract

We prove that for every integer $n \geq 2$ and every edge e of the complete graph K_n , the ordinary crossing number of $K_n - e$ is

$$\text{cr}(K_n - e) = \frac{1}{4} \left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor,$$

which was conjectured by Zheng, Lin, Yang and Cui in 2007. The proof relies on the September 2026 OpenAI preprint proving the Harary–Hill formula for $\text{cr}(K_n)$, whose stated results we use without re-proving them. Given that formula, the even orders follow from a theorem of Ouyang, Wang and Huang (2014), equivalently of Chia and Lee (2015), and the odd orders $n \leq 13$ follow from their counting bound and integrality. The new content is the lower bound for odd $n \geq 15$. We adapt the polynomial rank method of the OpenAI preprint to the missing edge. The adaptation restricts the test polynomials so that they put no weight on the missing pair. It pairs two different vanishing subspaces by mixed orthogonality. It also localizes, with a twisted detector that needs the full rectangular nonsingularity statement of the preprint, only at the vertices whose stars are complete. Via the product identity of Ouyang, Wang and Huang, the formula gives the crossing number of $K_m \square P_n$ conjectured by Zheng et al. for all $m \geq 4$ and $n \geq 1$, where P_n is the path with n edges. The new lower-bound input enters this consequence for odd $m \geq 13$ and $n \geq 2$; the literature comparison is bounded as stated in Section 7.

1 Introduction

An *ordinary drawing* of a finite simple graph places its vertices at distinct points of the plane and its edges as simple arcs joining their endpoints. Edge interiors avoid the vertices, and two edges meet in finitely many proper crossings, with no tangencies or triple points. The number $c(D)$ of crossings of a drawing D counts every crossing point, so two edges that meet several times contribute several crossings. The *crossing number* $\text{cr}(G)$ is the minimum of $c(D)$ over all ordinary drawings of G . No restriction is placed on vertex positions or edge routes. This is the convention of [7, Section 1], and all statements below use it.

Write

$$Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

The OpenAI preprint [7, Theorem 1.1] proves that $\text{cr}(K_n) = Z(n)$ for every $n \geq 3$, which is the Harary–Hill conjecture [5]. Our main result is the corresponding formula for the complete graph with one edge removed.

Theorem A. *For every integer $n \geq 2$ and every edge e of K_n ,*

$$\text{cr}(K_n - e) = Z(n) - \binom{(n-1)/2}{2} = \frac{1}{4} \left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor. \quad (1.1)$$

Equivalently, for every integer $s \geq 2$,

$$\text{cr}(K_{2s+1} - e) = \binom{s}{2}^2 - \binom{s}{2}, \quad \text{cr}(K_{2s} - e) = \binom{s}{2} \binom{s-1}{2} - \binom{s-1}{2} = \frac{(s^2-1)(s-2)^2}{4}. \quad (1.2)$$

The two expressions in (1.1) agree for every $n \geq 2$. For $n = 2s + 1$ the four floors are $s + 1, s, s - 1, s - 2$, and $\binom{s}{2}^2 - \binom{s}{2} = \frac{s(s-1)(s+1)(s-2)}{4}$. For $n = 2s$ they are $s + 1, s - 1, s - 2, s - 2$, and $Z(2s) - \binom{s-1}{2} = \frac{(s-1)(s-2)}{4}(s(s-1) - 2) = \frac{(s+1)(s-1)(s-2)^2}{4}$. All automorphisms of K_n act transitively on edges, so all graphs $K_n - e$ are isomorphic. The value is 0 for $n \leq 5$.

For a graph G and the path P_n with vertices $0, 1, \dots, n$ (so n edges, the convention of [8, Section 5]), $G \square P_n$ denotes the Cartesian product.

Corollary B. *For all integers $m \geq 4$ and $n \geq 1$,*

$$\text{cr}(K_m \square P_n) = \frac{1}{4} \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-2}{2} \right\rfloor \left(n \left\lfloor \frac{m+4}{2} \right\rfloor + \left\lfloor \frac{m-4}{2} \right\rfloor \right). \quad (1.3)$$

Corollary B is one application of Theorem A, through the identity $\text{cr}(K_m \square P_n) = (n - 1) \text{cr}(K_{m+2} - e) + 2 \text{cr}(K_{m+1})$ of Ouyang, Wang and Huang [8, Corollary 5.1]. We do not regard it as a separate result.

1.1 History and what is new

Zheng, Lin, Yang and Cui [10] gave the upper bound (1.1) for $\text{cr}(K_n - e)$, conjectured equality, and proved it for $n \leq 8$. In the same paper they proved $\text{cr}(K_6 \square P_n) = 15n + 3$, proved the lower bound $\text{cr}(K_m \square P_n) \geq (n - 1) \text{cr}(K_{m+2} - e) + 2 \text{cr}(K_{m+1})$, and conjectured (1.3). We were not able to obtain the text of [10]. These attributions follow Ouyang, Wang and Huang [8, Section 1, display (1.1)–(1.2), and Lemma 2.3] and the survey of Clancy, Haythorpe and Newcombe [4, Theorem 2.19, Conjecture 2.20, Theorems 3.12–3.13, Conjecture 3.14].

Ouyang, Wang and Huang [8] proved two lower bounds for $\text{cr}(K_m - e)$ in terms of complete-graph crossing numbers [8, Theorems 3.1 and 3.2]. They deduced the even case of the conjecture from the Harary–Hill formula for the preceding odd order [8, Theorem 3.3], and settled $m \leq 12$ using the then-known values of $\text{cr}(K_m)$ [8, Theorem 3.4]. Chia and Lee [3] also obtained equivalent results: the upper bound [3, Lemma 1], the counting lower bound [3, Theorem 1], the even case under the odd Harary–Hill hypothesis [3, Theorem 2], the values for $n \leq 12$ [3, Corollary 1], and the conjecture [3, Conjecture 1]. Ouyang, Wang and Huang also proved the product identity [8, Corollary 5.1] and deduced (1.3) for $m \leq 10$ [8, Corollary 5.2].

Once the Harary–Hill formula is available for every order [7], the following parts of Theorem A are *classical consequences* and are not claimed as new:

- the upper bound for every n [10, 3]; Section 5 recounts it in the two-page drawing of [7];
- the lower bound for every even n , by [8, Theorem 3.3] or [3, Theorem 2]; Section 4 reproduces the proof of [8, Theorem 3.2];
- the lower bound for odd $n \leq 13$, from [8, Theorem 3.1] (equivalently [3, Theorem 1]) and integrality; see Section 7.

The *new* part is the lower bound $\text{cr}(K_{2s+1} - e) \geq \binom{s}{2}^2 - \binom{s}{2}$ for $s \geq 7$, that is, for odd $n \geq 15$ (Proposition 3.7). There the counting bound of [8, Theorem 3.1] falls short of the target by exactly $\frac{(s-1)(s-2)}{2(2s+1)}$, which is at least 1 for every $s \geq 7$. Our proof of Proposition 3.7 covers every odd $n \geq 5$, but for $n \leq 13$ it only reproves known consequences. Correspondingly, the new lower-bound input enters Corollary B exactly for odd $m \geq 13$ and $n \geq 2$, subject to the bounded literature comparison in Section 7. At $n = 1$ the coefficient $n - 1$ vanishes, and the value $2Z(m + 1)$ already follows from Harary–Hill. The matching upper bound on the new range follows from the classical minus-edge upper bound, the product identity and Harary–Hill. For even m and odd $m \leq 11$, the value $\text{cr}(K_{m+2} - e)$ needed in [8, Corollary 5.1] was already available after the Harary–Hill formula.

1.2 Method and relation to other work

The OpenAI preprint [7] proves the odd lower bound $\text{cr}(K_{2s+1}) \geq \binom{s}{2}^2$ as follows. A skew bilinear form built from signed crossings and rotation orders vanishes on pairs of cycles. Weighting edges by values of interpolation-weighted polynomials yields cycles, hence polynomial identities. These identities make a space of polynomials of dimension $\binom{s}{2}^2$ embed isotropically into the coordinates indexed by crossing edge pairs.

For $K_n - e$ the same polynomial chains put nonzero weight on the missing pair. Four changes are needed, and they are the content of Section 3.

- (i) *Missing-pair tests* (Lemma 3.1). A test polynomial gives a valid identity whenever it satisfies two polynomial symmetry conditions at the missing pair. The conditions are sufficient, not necessary (Remark 3.2).
- (ii) *Split spaces and mixed orthogonality* (Lemmas 3.3 and 3.4). The full space of [7] is not admissible. We use two subspaces $\mathcal{S}_a, \mathcal{S}_b$, each of codimension $\binom{s}{2}$, vanishing on the missing pair in the first or in the second edge slot, and show that their images are orthogonal to each other. Their intersection would lose $\binom{s}{2} - 1$ further dimensions per side.
- (iii) *Twisted detector* (Lemma 3.5). The localization tests must carry extra factors $(y - t_a)$ and $(v - t_a)$. The resulting pairing is controlled by the *rectangular* nonsingularity in [7, Lemma 3.3], not by its square consequence.
- (iv) *Localization away from the missing edge* (Lemma 3.6). Only the $2s-1$ vertices with complete stars are used for localization. This still exceeds the degree $2s-4$ of the relevant diagonal polynomial, and the degree descent preserves the vanishing condition at the missing pair.

Sadhu [9] recently determined $\text{cr}(K_{m,n} - e)$ for complete bipartite graphs, a family also treated by Chia and Lee [3]. That argument is stated to be inspired by the same algebraic framework, in its bipartite form [6]. That preprint appeared before the present work. Its stated new ingredients are exact node-deletion maps, a balanced star-compression rank inequality with dimension losses, an extension of the bipartite order detector to unequal rectangles, and a signed three-edge completion that replaces the missing edge. In the passages we compared (Section 7), these operations differ from ours. Neither theorem implies the other. We therefore claim the specific argument for $K_n - e$, not the general idea of adapting the framework of [7] to a deleted edge.

Asiri, Burdett, Chimani, Haythorpe, Newcombe and Wagner [1] gave a general computational framework for $\text{cr}(G \square P_n)$ and applied it to graphs of order five and six. It does not treat complete graphs of arbitrary order.

1.3 Organization and status

Section 2 lists the exact upstream statements used. Section 3 proves the odd lower bound. Section 4 reproduces the even lower bound. Section 5 gives the upper bound and Section 6 assembles Theorem A and Corollary B. Section 7 records the novelty boundary, Section 8 the computational support, and Section 9 the audited algebraic support and remaining formalization gaps. The finite computations support the proof but are not part of it.

2 Conventions and upstream statements

We use the following statements from the OpenAI preprint [7], as fixed in its source at commit fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb of <https://github.com/openai/math>, file preprints/The-crossing-number-of-complete-graphs-September-23-2026/build/main.

tex (SHA-256 5ae2c607...2301). We cite them by their source labels and use them as stated. We do not re-prove them, and the correctness of our results depends on theirs.

Let G be a finite simple graph with a fixed orientation of each edge. Its real edge space $C_1(G)$ has the oriented edges as a basis; reversing an orientation negates the vector. For a vertex set $\{1, \dots, n\}$ we write $[ip]$ for the oriented pair from i to p , with $[pi] = -[ip]$ and $[ii] = 0$. The boundary map is $\partial[ip] = \mathbf{e}_p - \mathbf{e}_i$, and a *cycle* is an element of $\ker \partial$. Following [7, Section 2], a drawing is *normalized* if it is polygonal, straight near every vertex and crossing, and has no crossing between two edges sharing an endpoint. Independent edges may cross repeatedly.

Premise U1 ([7, Lemma A.1, label `lem:normalization`]). *For every finite simple graph, a minimum ordinary drawing can be chosen normalized.*

For a normalized drawing, let $I(e, f)$ be the sum of the signs of the interior crossings of two distinct oriented edges, with $I(e, e) = 0$. At each vertex w , cut the circular order of the outgoing spokes at a gap and list the incident edges counterclockwise. Put $T_w(e, f) = 1$ if e precedes f , -1 if f precedes e , and 0 if $e = f$. Put $\sigma_{we} = 1$ if e points out of w and -1 if it points into w . Define a bilinear form on $C_1(G)$ by

$$J(e, f) = I(e, f) + \frac{1}{2} \sum_{w \text{ a common endpoint of } e, f} \sigma_{we} \sigma_{wf} T_w(e, f) \quad (2.1)$$

[7, equation eq:J]. It is skew-symmetric.

Premise U2 ([7, Lemma 2.1, label `lem:cycles`]). *For a normalized drawing of any finite simple graph G , $J(z, z') = 0$ for any two cycles z, z' of G .*

The proof in [7] states explicitly that the argument applies to every finite simple graph; we apply it to $G = K_n - e$, never to K_n .

Premise U3 ([7, Lemma 3.1, label `lem:weights`]). *For distinct reals $u_1, \dots, u_N$ and $\Gamma(t) = \prod_{p=1}^N (t - u_p)$, one has $\sum_{p=1}^N A(u_p) / \Gamma'(u_p) = 0$ for every polynomial A of degree at most $N - 2$.*

For nonnegative integers α, β let $\mathcal{P}_{\alpha, \beta}$ be the real polynomials in (u, v) of degree at most α in u and at most β in v . Let $s \geq 2$ and let $u_1, \dots, u_{2s}$ be distinct reals carrying an arbitrary prescribed linear order, not necessarily their numerical order. Put $S(p, q) = \frac{1}{2}$ if p precedes q , $-\frac{1}{2}$ if q precedes p , $S(p, p) = 0$, $\Gamma(t) = \prod_{p=1}^{2s} (t - u_p)$, and

$$M(A, B) = \sum_{p, q=1}^{2s} S(p, q) \frac{u_p - u_q}{\Gamma'(u_p) \Gamma'(u_q)} A(u_p, u_q) B(u_p, u_q) \quad (2.2)$$

[7, equation eq:M].

Premise U4 ([7, Lemma 3.3, label `lem:detector`, first sentence]). *The pairing M on $\mathcal{P}_{s-1, s-2} \times \mathcal{P}_{s-2, s-1}$ is nonsingular.*

Premise U5 ([7, Theorem 1.1, label `thm:main`]). $\text{cr}(K_n) = Z(n)$ for every integer $n \geq 3$.

Premise U6 ([7, Section 5 and Proposition 5.1, labels `sec:upper`, `prop:upper`]). *Let $n \geq 3$ and $m = \lfloor n/2 \rfloor$. Place vertices $0, 1, \dots, n-1$ at increasing positions on a line, and draw the edge ij as a semicircle above the line if $(i+j) \bmod n < m$ and below it otherwise. For generic positions this is an ordinary drawing of K_n . Its crossings are exactly the pairs of edges on the same side whose four endpoints alternate along the line, each such pair crossing once, and it has exactly $Z(n)$ crossings.*

The bipartite companion [6] is not used. In [7], the results labelled `lem:tests`, `lem:injectivity` and `prop:odd` concern K_n . We do not use them as black boxes. Their proofs serve as templates for Lemmas 3.1, 3.6 and Proposition 3.7, and every changed step is proved below.

The external results used are as follows.

- [8, Corollary 5.1]: for $m \geq 4$ and $n \geq 1$, $\text{cr}(K_m \square P_n) = (n-1) \text{cr}(K_{m+2} \setminus e) + 2 \text{cr}(K_{m+1})$, with P_n the path with vertices $0, \dots, n$. It is unconditional. Its proof uses [8, Theorems 5.1, 5.2 and Lemma 4.1] and a lemma of Bokal [2]. We cite it and have not re-checked that chain.
- [8, Theorem 3.2] is reproduced with proof in Section 4. [8, Theorem 3.1] is used only in Section 7 to delimit the novelty.

Table 1 summarizes where each input enters.

Input	Source locator	Used in
U1 normalization	[7], <code>lem:normalization</code>	Odd (Sec. 3) and even (Sec. 4) lower bounds
U2 cycle pairing	[7], <code>eq:J</code> , <code>lem:cycles</code>	Lemma 3.1, applied to $G = K_n - e$
U3 interpolation weights	[7], <code>lem:weights</code>	Lemma 3.1 (boundary of test chains)
U4 rectangular detector	[7], <code>eq:M</code> , <code>lem:detector</code>	Lemma 3.5
U5 Harary–Hill formula	[7], <code>thm:main</code>	Even lower bound; Corollary B
U6 two-page drawing	[7], <code>sec:upper</code> , <code>prop:upper</code>	Upper bound (Sec. 5)
OWH Cor. 5.1	[8], Corollary 5.1 (cited)	Corollary B
OWH Thm. 3.2	[8], Theorem 3.2 (reproved)	Even lower bound (Sec. 4)
OWH Thm. 3.1	[8], Theorem 3.1	Novelty boundary only (Sec. 7)

Table 1: Dependencies. The truth of U1–U6 is taken from [7]; OWH Corollary 5.1 is cited from the published paper.

3 The odd lower bound

Throughout this section $s \geq 3$ and $n = 2s + 1$. Let $G = K_n - ab$ for two distinct vertices a, b of K_n , and fix a normalized minimum drawing D of G (Premise U1), so $c(D) = \text{cr}(G)$. Fix an orientation of each edge of G and let J be the form (2.1) on $C_1(G)$.

3.1 Zero extension and test chains

Regard $C_1(G)$ as the subspace of $C_1(K_n)$ spanned by all oriented pairs except $\pm[ab]$. Extend J to a bilinear form $\tilde{J}$ on $C_1(K_n)$ by setting $\tilde{J}(f, g) = 0$ whenever f or g is $[ab]$. Then $\tilde{J}$ is skew-symmetric and restricts to J on $C_1(G)$. For ordered vertex pairs we write $J([ip], [kq])$ for $\tilde{J}([ip], [kq])$; it vanishes when $i = p$, when $k = q$, or when $\{i, p\}$ or $\{k, q\}$ equals $\{a, b\}$.

Choose distinct reals $t_1, \dots, t_n$ and set

$$F(t) = \prod_{i=1}^n (t - t_i), \quad w_i = \frac{1}{F'(t_i)} \neq 0.$$

For a polynomial $H(x, u, y, v)$ write

$$\Sigma(H) = \sum_{i,p,k,q=1}^n J([ip], [kq]) w_i w_p w_k w_q H(t_i, t_p, t_k, t_q). \quad (3.1)$$

Lemma 3.1 (Missing-pair cycle tests). *Let $H(x, u, y, v)$ have degree at most $n - 2$ in each variable separately and satisfy the polynomial identities*

$$H(t_a, t_b, y, v) = H(t_b, t_a, y, v), \quad H(x, u, t_a, t_b) = H(x, u, t_b, t_a). \quad (3.2)$$

Then $\Sigma(H) = 0$.

Proof. Tensor decomposition. Let $\mathcal{V}$ be the space of polynomials $A(x, u)$ of degree at most $n - 2$ in each of x, u . Multiplication $A \otimes B \mapsto A(x, u)B(y, v)$ identifies $\mathcal{V} \otimes \mathcal{V}$ with the space of four-variable polynomials of separate degree at most $n - 2$, since it maps a monomial basis to a monomial basis. Let $\varphi(A) = A(t_a, t_b) - A(t_b, t_a)$. This linear functional is nonzero, since $\varphi(x) = t_a - t_b$. Choose $e_0 \in \mathcal{V}$ with $\varphi(e_0) = 1$ and a basis $e_1, \dots, e_N$ of $\mathcal{A} = \ker \varphi$. Write $H = \sum_{j,k=0}^N c_{jk} e_j(x, u) e_k(y, v)$. The first identity in (3.2) says $(\varphi \otimes \text{id})H = \sum_k c_{0k} e_k(y, v) = 0$, so $c_{0k} = 0$ for all k . The second says $(\text{id} \otimes \varphi)H = \sum_j c_{j0} e_j(x, u) = 0$, so $c_{j0} = 0$ for all j . Hence $H = \sum_r A_r(x, u) B_r(y, v)$ with all $A_r, B_r \in \mathcal{A}$.

Test chains are cycles of G . For $A \in \mathcal{V}$ put

$$z_A = \sum_{i,p=1}^n w_i w_p A(t_i, t_p) [ip] \in C_1(K_n).$$

Its coefficient on the basis vector $[ab]$ is $w_a w_b (A(t_a, t_b) - A(t_b, t_a)) = w_a w_b \varphi(A)$, because the two ordered terms carry opposite vectors $[ab]$ and $[ba] = -[ab]$. Thus $z_A \in C_1(G)$ whenever $A \in \mathcal{A}$. For a monomial $A = x^\alpha u^\beta$ with $\alpha, \beta \leq n - 2$,

$$\partial z_A = \left(\sum_i w_i t_i^\alpha \right) \sum_p w_p t_p^\beta \mathbf{e}_p - \left(\sum_p w_p t_p^\beta \right) \sum_i w_i t_i^\alpha \mathbf{e}_i = 0$$

by Premise U3 with $N = n$. By linearity $\partial z_A = 0$ for all $A \in \mathcal{V}$. The boundary map of G is the restriction of that of K_n , so z_A is a cycle of G for every $A \in \mathcal{A}$.

Conclusion. For $A, B \in \mathcal{A}$, bilinearity gives $\Sigma(A(x, u)B(y, v)) = \tilde{J}(z_A, z_B) = J(z_A, z_B)$, which is 0 by Premise U2 applied to G . Summing over r gives $\Sigma(H) = 0$. $\square$

Remark 3.2 (The conditions are sufficient only). The identities (3.2) are not necessary for $\Sigma(H) = 0$, and we make no claim that they characterize the vanishing tests. For example, $H = xy$ violates both identities, since $H(t_a, t_b, y, v) - H(t_b, t_a, y, v) = (t_a - t_b)y$ and $H(x, u, t_a, t_b) - H(x, u, t_b, t_a) = (t_a - t_b)x$. Yet $\Sigma(xy) = \tilde{J}(z_x, z_x) = 0$ by skew-symmetry, for every drawing. The proof below uses only the sufficient direction.

3.2 Support coordinates and split spaces

Let $\mathcal{T}$ be the set of ordered pairs (f, g) of (undirected) edges of G with no common endpoint and $J(f, g) \neq 0$. The condition does not depend on the chosen orientations, and $\mathcal{T}$ is closed under $(f, g) \mapsto (g, f)$ because J is skew. For independent edges the correction term in (2.1) is empty, so $J(f, g) = I(f, g) \neq 0$ forces f and g to cross. Each crossing involves exactly two edges, so

$$c(D) \geq \frac{|\mathcal{T}|}{2}. \quad (3.3)$$

Repeated crossings only increase $c(D)$, and cancellation of signs only removes pairs from $\mathcal{T}$.

Let $\mathcal{S}$ be the space of polynomials $P(x, u, y, v)$ of degree at most $s - 2$ in each variable that are symmetric under $x \leftrightarrow u$ and under $y \leftrightarrow v$ (exchanging the two pairs is not required). Let Sym be the space of symmetric polynomials $q(x, u)$ of degree at most $s - 2$ in each variable. It has the basis $x^\alpha u^\beta + x^\beta u^\alpha$ ($\alpha < \beta$), $x^\alpha u^\alpha$, with $0 \leq \alpha \leq \beta \leq s - 2$, so

$$d := \dim \text{Sym} = \binom{s}{2}, \quad \mathcal{S} = \text{Sym} \otimes \text{Sym}, \quad \dim \mathcal{S} = d^2$$

[7, equation eq:dim]. Let $\psi : \text{Sym} \rightarrow \mathbb{R}$ be evaluation at (t_a, t_b) . It is nonzero, as $\psi(1) = 1$. Define

$$\begin{aligned}\mathcal{S}_a &= \{P \in \mathcal{S} : P(t_a, t_b, y, v) \equiv 0\} = \ker(\psi \otimes \text{id}), \\ \mathcal{S}_b &= \{Q \in \mathcal{S} : Q(x, u, t_a, t_b) \equiv 0\} = \ker(\text{id} \otimes \psi).\end{aligned}$$

By the within-pair symmetries, every $P \in \mathcal{S}_a$ also satisfies $P(t_b, t_a, y, v) \equiv 0$, and every $Q \in \mathcal{S}_b$ satisfies $Q(x, u, t_b, t_a) \equiv 0$.

Lemma 3.3 (Dimensions). $\mathcal{S}_a = \ker \psi \otimes \text{Sym}$ and $\mathcal{S}_b = \text{Sym} \otimes \ker \psi$. Hence $\dim \mathcal{S}_a = \dim \mathcal{S}_b = d^2 - d = \binom{s}{2}^2 - \binom{s}{2}$. Moreover $\mathcal{S}_a \cap \mathcal{S}_b = \ker \psi \otimes \ker \psi$ has dimension $(d-1)^2$.

Proof. Choose a basis $f_0, \dots, f_{d-1}$ of Sym with $\psi(f_0) = 1$ and $f_1, \dots, f_{d-1} \in \ker \psi$. Writing $P = \sum c_{jk} f_j \otimes f_k$, we get $(\psi \otimes \text{id})P = \sum_k c_{0k} f_k$, which vanishes exactly when the row $j = 0$ is zero. The same applies to the column $k = 0$ for $\mathcal{S}_b$, and to both together for the intersection. $\square$

The evaluation map

$$E : \mathcal{S} \rightarrow \mathbb{R}^{\mathcal{T}}, \quad E(P)_{(\{i,p\}, \{k,q\})} = P(t_i, t_p, t_k, t_q),$$

is well defined by the within-pair symmetries.

Lemma 3.4 (Mixed orthogonality). Let $\Delta = (x-u)(y-v)(x-y)(x-v)(u-y)(u-v)$. For $\alpha = (\{i, p\}, \{k, q\}) \in \mathcal{T}$ put

$$a_\alpha = J([ip], [kq]) w_i w_p w_k w_q \Delta(t_i, t_p, t_k, t_q),$$

which does not depend on the orientations $[ip], [kq]$ chosen. Every a_α is nonzero, and

$$\sum_{\alpha \in \mathcal{T}} a_\alpha E(P)_\alpha E(Q)_\alpha = 0 \quad (P \in \mathcal{S}_a, Q \in \mathcal{S}_b). \quad (3.4)$$

Proof. Reversing $[ip]$ negates J and exchanges x, u in Δ . That exchange negates $(x-u)$, swaps $(x-y)$ with $(u-y)$ and swaps $(x-v)$ with $(u-v)$, so it negates Δ . The same holds for $[kq]$. Hence a_α is well defined. It is nonzero because $J \neq 0$ on $\mathcal{T}$, all $w_i \neq 0$, and the four labels are distinct.

Let $H = PQ\Delta$. The factor Δ has degree 3 in each variable, so H has degree at most $2(s-2) + 3 = 2s-1 = n-2$ in each variable. Both sides of the first identity in (3.2) vanish because $P \in \mathcal{S}_a$, and both sides of the second vanish because $Q \in \mathcal{S}_b$. Lemma 3.1 gives $\Sigma(H) = 0$. We sort the terms of (3.1). Terms with $i = p$ or $k = q$ vanish since $[ii] = 0$. Terms in which $\{i, p\}$ or $\{k, q\}$ is the missing pair vanish by the zero extension. Terms in which the two pairs share a vertex vanish since Δ does. The remaining terms have four distinct indices and correspond to ordered pairs α of independent edges of G , each with four choices of orientations. For each choice the term equals $a_\alpha E(P)_\alpha E(Q)_\alpha$: the orientation signs of J and Δ cancel as above, and P, Q are symmetric within pairs. Pairs outside $\mathcal{T}$ have $J = 0$. Hence $0 = \Sigma(H) = 4 \sum_{\alpha \in \mathcal{T}} a_\alpha E(P)_\alpha E(Q)_\alpha$. $\square$

The form $\langle X, Y \rangle = \sum_{\alpha \in \mathcal{T}} a_\alpha X_\alpha Y_\alpha$ on $\mathbb{R}^{\mathcal{T}}$ is symmetric and diagonal with nonzero entries, hence nondegenerate. It is not the skew form J . Exchanging the two edges of α gives a different coordinate whose coefficient is $-a_\alpha$, because J is skew while $\Delta(y, v, x, u) = \Delta(x, u, y, v)$. This does not affect nondegeneracy. Lemma 3.4 says that $E(\mathcal{S}_b) \subseteq E(\mathcal{S}_a)^\perp$. Neither image needs to be isotropic.

3.3 The twisted detector

Lemma 3.5 (Twisted detector). *Let $s \geq 2$, let $u_1, \dots, u_{2s}$ be distinct reals with any prescribed linear order, let M be the pairing (2.2), and let $c \in \{1, \dots, 2s\}$. If $A \in \mathcal{P}_{s-2, s-2}$ satisfies $M(A, (v - u_c)B') = 0$ for every $B' \in \mathcal{P}_{s-1, s-2}$, then $A = 0$.*

Proof. The value $M(A, B)$ depends only on the product AB . Hence $M(A, (v - u_c)B') = M(B', \tilde{A})$ with $\tilde{A} = (v - u_c)A \in \mathcal{P}_{s-2, s-1}$. By Premise U4, the pairing on $\mathcal{P}_{s-1, s-2} \times \mathcal{P}_{s-2, s-1}$ is nonsingular. Since $M(B', \tilde{A}) = 0$ for every $B' \in \mathcal{P}_{s-1, s-2}$, we get $\tilde{A} = 0$. As $\mathbb{R}[u, v]$ is an integral domain and $v - u_c \neq 0$, $A = 0$. $\square$

The proof does not claim that multiplication by $v - u_c$ is invertible on node values; it is not, since it vanishes at the nodes with $v = u_c$. The lemma uses the nonsingularity of the full rectangular pairing. The square consequence [7, equation eq:detector], which tests $A \in \mathcal{P}_{s-2, s-2}$ against all of $\mathcal{P}_{s-1, s-1}$, does not apply directly, because the twisted test functions $(v - u_c)B'$ form a proper subspace of $\mathcal{P}_{s-1, s-1}$.

3.4 Injectivity

Lemma 3.6 (Injectivity). *The restrictions of E to $\mathcal{S}_a$ and to $\mathcal{S}_b$ are injective.*

Proof. Let $P \in \mathcal{S}_a$ with $E(P) = 0$. The proof has three steps.

Step 1: localization at complete stars. Let $U(x)$ have degree at most s , $V'(y)$ degree at most $s - 1$, and $B' \in \mathcal{P}_{s-1, s-2}$ (in the variables u, v). Consider

$$H = P(x, u, y, v) (u - v)(u - y)(v - x) U(x) (y - t_a) V'(y) (v - t_a) B'(u, v). \quad (3.5)$$

Its separate degrees are at most

$$\begin{aligned} x : & (s - 2) + 1 + s &= 2s - 1, \\ u : & (s - 2) + 2 + (s - 1) &= 2s - 1, \\ y : & (s - 2) + 1 + 1 + (s - 1) &= 2s - 1, \\ v : & (s - 2) + 2 + 1 + (s - 2) &= 2s - 1, \end{aligned}$$

where the middle terms count, in order, the factors $(v - x)$; $(u - v)$, $(u - y)$; $(u - y)$, $(y - t_a)$; and $(u - v)$, $(v - x)$, $(v - t_a)$. All are at most $n - 2$. Both sides of the first identity in (3.2) vanish because $P \in \mathcal{S}_a$. For the second, $H(x, u, t_a, t_b) = 0$ through the factor $(y - t_a)$ and $H(x, u, t_b, t_a) = 0$ through the factor $(v - t_a)$. Lemma 3.1 gives $\Sigma(H) = 0$.

We evaluate $\Sigma(H)$. Terms with two independent edges of G contribute $J \cdot P(\dots)$, which vanishes either because $J = 0$ or because the pair lies in $\mathcal{T}$ and $E(P)$ vanishes there. Terms involving the missing pair or a loop vanish. The factor $(u - v)(u - y)(v - x)$ kills the coincidences $p = q$, $p = k$ and $q = i$. The only surviving terms have $i = k$ with i, p, q distinct, and there

$$(u - v)(u - y)(v - x)|_{(t_i, t_p, t_i, t_q)} = (t_p - t_q)(t_p - t_i)(t_q - t_i).$$

With $F_i(t) = \prod_{j \neq i} (t - t_j)$ we have $F'(t_p) = (t_p - t_i)F'_i(t_p)$ for $p \neq i$, so $w_p(t_p - t_i) = 1/F'_i(t_p)$. Therefore

$$0 = \Sigma(H) = \sum_{i=1}^n w_i^2 U(t_i)(t_i - t_a) V'(t_i) \gamma_i(B'), \quad (3.6)$$

where

$$\gamma_i(B') = \sum_{p, q \neq i} J([ip], [iq]) \frac{t_p - t_q}{F'_i(t_p)F'_i(t_q)} P(t_i, t_p, t_i, t_q) (t_q - t_a) B'(t_p, t_q).$$

Let $i \notin \{a, b\}$. Every pair $\{i, p\}$ with $p \neq i$ is then an edge of G . For $p \neq q$ the edges ip, iq share only the endpoint i , so they do not cross (the drawing is normalized), and both

orientations $[ip], [iq]$ point out of i . By (2.1), $J([ip], [iq]) = \frac{1}{2}T_i(ip, iq) =: S_i(p, q)$, the sign function of a linear order on the $2s$ indices $p \neq i$; the diagonal terms vanish because of the factor $t_p - t_q$. Hence $\gamma_i(B') = M_i(P(t_i, u, t_i, v), (v - t_a)B'(u, v))$, where M_i is the pairing (2.2) on the $2s$ nodes $\{t_p : p \neq i\}$ with this order and $\Gamma = F_i$. For $i \in \{a, b\}$ the star of i in G is incomplete; we do not interpret γ_a, γ_b at all.

Isolation. Fix B' and $i_0 \notin \{a, b\}$. Identity (3.6) holds for every U and V' , and it is linear in their product. The products $U(t)V'(t)$ with $\deg U \leq s$ and $\deg V' \leq s - 1$ span all polynomials of degree at most $2s - 1$, because every exponent $k \leq 2s - 1$ splits as $k_1 + k_2$ with $k_1 \leq s$, $k_2 \leq s - 1$. So we may replace $U(t)(t - t_a)V'(t)$ by any polynomial of degree at most $2s$ that vanishes at t_a . The Lagrange polynomial L_{i_0} , of degree $n - 1 = 2s$, equal to 1 at t_{i_0} and 0 at all other nodes, is such a polynomial because $a \neq i_0$. It also vanishes at t_b . Substituting it into (3.6) kills every term except $i = i_0$, including the uninterpreted terms $i = a, b$, and leaves $w_{i_0}^2 \gamma_{i_0}(B') = 0$. As $w_{i_0} \neq 0$,

$$M_{i_0}(P(t_{i_0}, u, t_{i_0}, v), (v - t_a)B') = 0 \quad \text{for every } B' \in \mathcal{P}_{s-1, s-2}.$$

Here $P(t_{i_0}, u, t_{i_0}, v) \in \mathcal{P}_{s-2, s-2}$, and t_a is one of the $2s$ nodes $\{t_p : p \neq i_0\}$. Lemma 3.5 with $u_c = t_a$ gives $P(t_{i_0}, u, t_{i_0}, v) = 0$ as a polynomial, for each of the $2s - 1$ indices $i_0 \notin \{a, b\}$.

Step 2: collision factors. The polynomial $P(x, u, x, v)$ has degree at most $2s - 4$ in x . Each of its coefficients in (u, v) vanishes at the $2s - 1 > 2s - 4$ distinct values t_{i_0} , $i_0 \notin \{a, b\}$, so $P(x, u, x, v) = 0$. Viewing P as a polynomial in y over $\mathbb{R}[x, u, v]$, we get $(x - y) \mid P$. Applying the symmetries $x \leftrightarrow u$ and $y \leftrightarrow v$ gives divisibility by $u - y$, $x - v$ and $u - v$ as well. Each of these four linear forms is prime in $\mathbb{R}[x, u, y, v]$, since the quotient by it is a polynomial ring, and they are pairwise nonassociate. Since the ring is a unique factorization domain, their product

$$C = (x - y)(x - v)(u - y)(u - v)$$

divides P .

Step 3: descent. We have shown that C divides every element of $\ker E \cap \mathcal{S}_a$. Suppose this kernel contains a nonzero element and choose one, P , of minimum degree in x . Write $P = CR$ with $R \neq 0$. Since C has degree exactly 2 in each variable, R has degree $\deg_z P - 2$ in each variable z . In particular every separate degree of P is at least 2. This is already impossible when $s = 3$, where all separate degrees are at most 1. For $s \geq 4$, R has separate degrees at most $s - 4 \leq s - 2$. The polynomial C is invariant under both within-pair swaps; applying a swap to $P = CR$ and cancelling C in the integral domain shows that R is symmetric within pairs. So $R \in \mathcal{S}$. Next,

$$0 = P(t_a, t_b, y, v) = C(t_a, t_b, y, v) R(t_a, t_b, y, v),$$

where $C(t_a, t_b, y, v) = (t_a - y)(t_a - v)(t_b - y)(t_b - v) \neq 0$ in $\mathbb{R}[y, v]$. This is a nonzero polynomial, although it vanishes at some points, so $R(t_a, t_b, y, v) \equiv 0$ and $R \in \mathcal{S}_a$. Finally C is nonzero at every evaluated pair of independent edges, since the four labels are distinct, so $E(R) = 0$. Thus R is a nonzero element of $\ker E \cap \mathcal{S}_a$ of smaller x -degree, a contradiction. Hence $E|_{\mathcal{S}_a}$ is injective.

The second slot. For $Q \in \mathcal{S}_b$ put $Q^{\text{sw}}(x, u, y, v) = Q(y, v, x, u)$. Then $Q^{\text{sw}} \in \mathcal{S}_a$, and $E(Q)_{(f, g)} = E(Q^{\text{sw}})_{(g, f)}$ for all $(f, g) \in \mathcal{T}$. Since $\mathcal{T}$ is closed under exchanging its two edges, $E(Q) = 0$ implies $E(Q^{\text{sw}}) = 0$, hence $Q^{\text{sw}} = 0$ and $Q = 0$. $\square$

3.5 The bound

Proposition 3.7 (Odd lower bound). *For every $s \geq 2$, every ordinary drawing of $K_{2s+1} - e$ has at least $\binom{s}{2}^2 - \binom{s}{2}$ crossings.*

Proof. For $s = 2$ the bound is 0. Let $s \geq 3$ and use the setting of this section. Put $W_a = E(\mathcal{S}_a)$ and $W_b = E(\mathcal{S}_b)$. By Lemma 3.4, $W_b \subseteq W_a^\perp$ for the nondegenerate form $\langle \cdot, \cdot \rangle$ on $V = \mathbb{R}^{\mathcal{T}}$. Nondegeneracy identifies V with V^* . Restriction of functionals from V to W_a is onto W_a^* , and its kernel corresponds to $W_a^\perp$, so $\dim W_a^\perp = |\mathcal{T}| - \dim W_a$. This holds whether or not the form is nondegenerate on W_a . Therefore $\dim W_a + \dim W_b \leq |\mathcal{T}|$. By Lemmas 3.6 and 3.3, the left side is $2\left(\binom{s}{2}^2 - \binom{s}{2}\right)$. With (3.3),

$$\text{cr}(K_{2s+1} - e) = c(D) \geq \frac{|\mathcal{T}|}{2} \geq \binom{s}{2}^2 - \binom{s}{2}.$$

Every ordinary drawing has at least $c(D)$ crossings. $\square$

Remark 3.8 (Why two spaces). For $P, Q \in \mathcal{S}$ the polynomial $PQ\Delta$ generally violates (3.2), so the full space of [7] cannot be used. A single admissible space for both slots would be $\mathcal{S}_a \cap \mathcal{S}_b$. By Lemma 3.3 it has dimension $(d-1)^2$, and its self-orthogonality would give only $\text{cr} \geq (d-1)^2$, short of the target $d^2 - d$ by $d-1 = \binom{s}{2} - 1$. Pairing two different spaces of codimension d gives exactly the conjectured value.

4 The even lower bound

The following is [8, Theorem 3.2] specialized to even order with Premise U5. Equivalent statements are [8, Theorem 3.3] and [3, Theorem 2]. We include the proof so that the paper is complete and to record that the local counts m_i below cancel exactly, so no bound on them is needed. This observation stays within the argument of [8].

Proposition 4.1. *For every $M \geq 3$, every ordinary drawing of $K_{2M} - e$ has at least $\frac{1}{4}(M^2 - 1)(M - 2)^2$ crossings.*

Proof. Let $n = 2M \geq 6$, let x, y be the ends of the missing edge and $v_1, \dots, v_{n-2}$ the other vertices, and let D be a normalized minimum drawing of $K_n - xy$ (Premise U1). Let K^x and K^y be the subdrawings obtained by deleting y and x , respectively. Both are drawings of K_{n-1} , so by Premise U5

$$c(K^x) = Z(n-1) + l_1, \quad c(K^y) = Z(n-1) + l_2, \quad l_1, l_2 \geq 0.$$

Crossing count by type. Call the edges among the v_i the core, and the edges at x and at y the two stars. For each i let a_i be the number of crossings on the two edges xv_i and v_iy ; they do not cross each other. Every crossing of D is between independent edges. A core-core crossing is counted twice in $c(K^x) + c(K^y)$ and not in $\sum a_i$. A crossing between a star edge and a core edge is counted once in $c(K^x) + c(K^y)$ and once in $\sum a_i$. A crossing between xv_i and yv_j ($i \neq j$) is counted in neither subdrawing and twice in $\sum a_i$, once in a_i and once in a_j . Two edges of the same star share an endpoint and do not cross. Hence

$$2c(D) = c(K^x) + c(K^y) + \sum_{i=1}^{n-2} a_i. \quad (4.1)$$

Restoring the edge. For each i , draw a new arc from x to y that runs alongside xv_i , passes around v_i inside a small disk about v_i , and then runs alongside v_iy . A sufficiently close parallel track follows the polygonal edges through their crossing neighbourhoods. It meets every other edge once for each crossing of that edge with xv_i or v_iy , and it does not meet xv_i , v_iy or itself. Inside the disk it crosses some number $m_i \geq 0$ of the other spokes at v_i . Generic local choices avoid vertices, tangencies and triple points. The result is an ordinary drawing D_i of K_n with

$$c(D_i) = c(D) + a_i + m_i = Z(n) + c_i, \quad c_i \geq 0$$

by Premise U5. The drawing D_i may have crossings between adjacent edges (for instance between the new edge and an edge at y that crossed xv_i); this does not affect the lower bound $c(D_i) \geq Z(n)$. Summing over i and eliminating $\sum a_i$ with (4.1) gives

$$nc(D) = 2Z(n-1) + l_1 + l_2 + (n-2)Z(n) + \sum_i c_i - \sum_i m_i. \quad (4.2)$$

Responsibilities. Let $r_D(v)$ be the number of crossings on edges incident to v . In D each crossing has four distinct endpoints, so $\sum_v r_D(v) = 4c(D)$. The crossings of D that involve no edge at x are exactly those of K^y , and each other crossing involves exactly one edge at x ; so $r_D(x) = c(D) - c(K^y)$, and similarly $r_D(y) = c(D) - c(K^x)$. In D_i the old spokes at v_i still do not cross each other, the new edge is not incident to v_i , and the only new crossings on spokes at v_i are the m_i local ones. Edges crossing xv_i or v_iy in D are not incident to v_i . Hence

$$r_D(v_i) + m_i = r_{D_i}(v_i) = c(D_i) - c(D_i - v_i) \leq Z(n) + c_i - Z(n-1),$$

because $D_i - v_i$ is a drawing of K_{n-1} . Summing,

$$4c(D) \leq 2c(D) - 2Z(n-1) - l_1 - l_2 + (n-2)(Z(n) - Z(n-1)) + \sum_i c_i - \sum_i m_i,$$

that is,

$$2c(D) \leq (n-2)Z(n) - nZ(n-1) - (l_1 + l_2) + \sum_i c_i - \sum_i m_i. \quad (4.3)$$

Elimination. By (4.2), $\sum c_i - \sum m_i = nc(D) - (n-2)Z(n) - 2Z(n-1) - l_1 - l_2$. Substituting into (4.3) gives

$$(n-2)c(D) \geq (n+2)Z(n-1) + 2(l_1 + l_2) \geq (n+2)Z(n-1).$$

The m_i have cancelled. With $Z(2M-1) = \frac{1}{4}(M-1)^2(M-2)^2$,

$$c(D) \geq \frac{2M+2}{2M-2} \cdot \frac{(M-1)^2(M-2)^2}{4} = \frac{(M^2-1)(M-2)^2}{4}. \quad \square$$

5 The upper bound

The upper bound is due to Zheng et al. [10] (as quoted in [8, Lemma 2.3]), and Chia and Lee give a cylindrical construction [3, Lemma 1]. We verify it in the drawing of Premise U6.

Proposition 5.1. *For every $n \geq 4$ there is an ordinary drawing of $K_n - e$ with exactly $Z(n) - \binom{\lfloor (n-1)/2 \rfloor}{2}$ crossings.*

Proof. Take the drawing of Premise U6 and delete the edge $\{0, t\}$, $t = \lfloor (n-1)/2 \rfloor$. The result is an ordinary drawing of $K_n - e$, whose crossings are those of K_n not involving $\{0, t\}$. An edge $\{i, j\}$ crosses $\{0, t\}$ exactly when it lies on the same side and its endpoints alternate with $0 < t$, that is, $0 < i < t < j \leq n-1$.

If $n = 2s+1$, then $m = s$ and $t = s$. The residue of $0+s$ is $s \geq m$, so $\{0, s\}$ lies below, where the residues are $s, \dots, 2s$. For $1 \leq i \leq s-1$ and $s+1 \leq j \leq 2s$ we have $s+2 \leq i+j \leq 3s-1$. If $i+j \leq 2s$ the residue is $i+j \in [s+2, 2s]$ (below); otherwise it is $i+j - (2s+1) \in [0, s-2]$ (above). The crossing edges are those with $j \leq 2s-i$, and their number is $\sum_{i=1}^{s-1} (s-i) = \binom{s}{2}$.

If $n = 2s$, then $m = s$ and $t = s-1$. The residue $s-1 < m$ puts $\{0, s-1\}$ above, where the residues are $0, \dots, s-1$. For $1 \leq i \leq s-2$ and $s \leq j \leq 2s-1$ we have $s+1 \leq i+j \leq 3s-3$. If $i+j \leq 2s-1$ the residue is $i+j \geq s+1$ (below); otherwise it is $i+j - 2s \in [0, s-3]$ (above). The crossing edges are those with $j \geq 2s-i$, and their number is $\sum_{i=1}^{s-2} i = \binom{s-1}{2}$.

In both cases the number of crossings on $\{0, t\}$ is $\binom{t}{2}$, and the remaining drawing has $Z(n) - \binom{t}{2}$ crossings. $\square$

6 Proofs of the main results

Proof of Theorem A. By edge-transitivity we may fix the missing edge. For $n = 2, 3$ the graph $K_n - e$ is a forest and the formula gives 0. For $n \geq 4$ Proposition 5.1 gives the upper bound. For the lower bound, odd $n = 2s + 1 \geq 5$ is Proposition 3.7, $n = 4$ has target 0, and even $n = 2M \geq 6$ is Proposition 4.1. The parity forms (1.2) and the floor form were matched in Section 1. $\square$

Proof of Corollary B. By [8, Corollary 5.1], for $m \geq 4$ and $n \geq 1$, $\text{cr}(K_m \square P_n) = (n - 1) \text{cr}(K_{m+2} - e) + 2 \text{cr}(K_{m+1})$. We insert Theorem A and Premise U5.

If $m = 2k$ ($k \geq 2$), then $\text{cr}(K_{2k+2} - e) = \frac{1}{4}k(k+2)(k-1)^2$ and $\text{cr}(K_{2k+1}) = \frac{1}{4}k^2(k-1)^2$, so

$$\text{cr}(K_m \square P_n) = \frac{k(k-1)^2}{4}((n-1)(k+2) + 2k) = \frac{k(k-1)^2}{4}(n(k+2) + k-2).$$

The floors in (1.3) are $k, k-1, k-1, k+2, k-2$.

If $m = 2k+1$ ($k \geq 2$), then $\text{cr}(K_{2k+3} - e) = \frac{1}{4}(k+2)(k+1)k(k-1)$ and $\text{cr}(K_{2k+2}) = \frac{1}{4}(k+1)k^2(k-1)$, so

$$\text{cr}(K_m \square P_n) = \frac{(k+1)k(k-1)}{4}((n-1)(k+2) + 2k) = \frac{(k+1)k(k-1)}{4}(n(k+2) + k-2).$$

The floors in (1.3) are $k+1, k, k-1, k+2, k-2$. For $n = 1$ the minus-edge term drops out and the value is $2Z(m+1)$; for $m = 4$ the value is $2n$. $\square$

7 Novelty boundary

Routine ranges. For odd $n = 2s + 1$, inserting Premise U5 into [8, Theorem 3.1],

$$\text{cr}(K_n - e) \geq \frac{1}{n} \left(2Z(n-1) + (n-2)(Z(n) - \lfloor \frac{n-3}{2} \rfloor) \right),$$

gives a bound exactly $\frac{(s-1)(s-2)}{2(2s+1)}$ below the target. Since $\frac{(s-1)(s-2)}{2(2s+1)} - 1 = \frac{s(s-7)}{2(2s+1)}$, the deficit is less than 1 exactly for $s \leq 6$. So integrality closes $n \leq 13$ (for example the bound is 2720/13 at $n = 13$, with ceiling 210) and fails for every odd $n \geq 15$ (at $n = 15$ the ceiling is 419, the target 420). Other classical routes also stay below the target for odd $n \geq 15$. Evaluating [8, Theorem 3.2] at odd order leaves a gap of $\frac{s(s-1)(s-2)}{2(2s-1)}$. Averaging over vertex deletions, charging $Z(2s)$ for the two ends of the missing edge and the even value of Theorem A for the others, leaves a gap of $\frac{(s-1)^2(s-2)}{2(2s-3)}$. These identities are checked exactly in Section 8. They delimit what these particular routes give; they are not a proof that no simpler argument exists.

Status before this work. The surveys [4] (Section 2.2.1 and Section 3.2) list the minus-edge conjecture as known for $n \leq 12$ and the product conjecture as confirmed for $m \leq 10$. The OpenAI preprint [7] proves only $\text{cr}(K_n)$; its companion [6] treats $\text{cr}(K_{m,n})$. Neither states a result on $K_n - e$ or on Cartesian products; we checked the statements, the final sections, and the README files at the cited commit.

Comparison with Sadhu's bipartite theorem. Sadhu [9, Theorem 1.1] proves an exact formula, written there as $\text{cr}(K_{m,n} - e) = Z(m, n) - t_m t_n$, for all positive m, n and every edge e , and states that the argument does not assume the Zarankiewicz formula. We read the preprint in the text rendering of its ResearchGate page, because the PDF link was not accessible to us. The passages compared are the abstract, the contribution paragraph, the balanced compression inequality (Lemma 3.2) and the beginning of its proof, the rectangular bipartite detector

(Lemma 4.1), the signed three-edge completion of the missing edge in the proof of Theorem 1.1, the closing rank comparison, and the closing questions. In those passages the missing edge is replaced by a signed combination of three present edges and the intersection identity is run on a completed matrix. A multi-index compression inequality absorbs the dimension losses. Our argument instead forces the test chains to have zero coefficient on the missing pair, pairs two different vanishing subspaces of one four-variable space, quotes the already proved rectangular nonsingularity for complete graphs, and never localizes at the incomplete stars. We have not re-proved Sadhu’s lemmas, and our comparison is limited to the passages listed.

Limits. Our literature search was bounded: the surveys above, the primary papers [8, 3, 1], and targeted web searches in October 2026. Within that search we found no earlier proof of the odd case $n \geq 15$ or of the matching product lower bound for odd $m \geq 13$ and path length $n \geq 2$. These bounded findings are not a priority certificate. The case $n = 1$ of the product and its matching upper bound on the stated range are not claimed as new.

8 Computational support

The proofs above do not depend on any computation. The accompanying package contains finite checks that test identities, bookkeeping and hypotheses used in the proof. They use several kinds of arithmetic, listed in Table 2. Not all of them are exact rational computations.

Check	Content	Arithmetic
R1–R5	Closed forms ($n \leq 3000$), two-page deletion count ($n \leq 56$), even identity, routine-range gaps (odd $n \leq 401$), Corollary B algebra ($m \leq 400$, $n \leq 12$)	Exact integers (integer division only) and rationals; the two-page count is combinatorial
R6	Rectangular detector instances and twisted detector of Lemma 3.5, $s \leq 5$, random orders, every twist node	Exact rational ranks
R7	$\dim \mathcal{S}_a = \dim \mathcal{S}_b = d^2 - d$, $\dim(\mathcal{S}_a \cap \mathcal{S}_b) = (d - 1)^2$, $s \leq 6$	Exact rational ranks
R8	Degree budgets and the identities (3.2) for the tests of Lemmas 3.4 and 3.6, $3 \leq s \leq 7$; enforced controls showing each twist factor is needed and the untwisted test of [7] is inadmissible	Exact polynomial identities over $\mathbb{Q}$
R9	Four-orientation factor and the non-converse $H = xy$, for random skew forms (not drawings), $n = 7, 9$	Exact rationals
Author checker	Lemma 3.1, orthogonality and ranks on two-page and random straight-line drawings, $n \leq 12$	Rationals for identities; floating-point coordinates and angles for straight-line geometry; ranks modulo $2^{61} - 1$
Author mod- p checker	Ranks and orthogonality for $13 \leq n \leq 19$ (retained log); $n = 15, 17, 19$ re-executed for the original package	NumPy arithmetic modulo $2^{31} - 1$
Reviewer checker	Independent controls, including $n = 15$	Exact integer geometry, rationals, ranks modulo 1000003

Table 2: Finite checks. R1–R9 are new for this package; the other three are unchanged copies of earlier internal scripts.

The release runner fails if any check fails. This matters for the author’s mod- p script, which prints a failure message but still exits with status zero. The runner treats that message as a

failure, and a self-test confirms it. The author’s stdlib checker prints the values of tests that violate (3.2) but does not assert anything about them; the enforced negative controls are R8 and R9. Early runs of the independent reviewer’s checker failed because the harness wrongly required certain invalid tests to be nonzero. For example, at $n = 15$ the first chosen missing edge had no crossings in the full straight-line drawing. These were faulty expectations in the harness, not counterexamples. The final checker distinguishes them, and its retained logs are cited in the package rather than reproduced here. No finite check establishes the theorem for all n .

9 Formal verification status

A separate Lean package has an independently audited algebraic-support result, recorded as `PASS_SCOPED_SUPPORT`. Its core contains 47 theorems and 17 definitions (64 declarations), with 9 author control theorems and 16 independent control theorems. The accompanying `formal-scope.json` identifies the package, review hash and coverage. This is not a machine-checked proof of Theorem A or Corollary B.

The checked support includes the tensor-kernel decomposition, the first-slot dimension $d^2 - d$, the weighted diagonal form, degree and missing-pair identities for $PQ\Delta$, the twisted rectangular detector, the local-test degree budget, Lagrange isolation, the root count $2s - 1 > 2s - 4$, and generic cancellation and descent tools. Rectangular nonsingularity (Premise U4) is an explicit theorem hypothesis in Lean, not a newly introduced axiom or a formal proof of that upstream result.

The drawing topology, cycle boundaries, local drawing sums and their identification with global complete-star nodes remain paper-only. So do the concrete diagonal substitutions and four-prime divisibility, instantiation of descent on the actual evaluation kernel, both evaluation injections (including the slot swap), the factor of four and the crossing-count comparison, the even responsibility argument, the upper drawing, all-order assembly and Corollary B. Generic support lemmas do not supply these missing bridges. The present revision reports the separate audit; it does not claim a new Lean compilation or a full crossing certificate.

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This work was carried out with substantial assistance from AI systems, and the author takes responsibility for its content. The proof of the odd case was first drafted by an AI research agent (Claude Opus 5.5, run in the Factory Droid harness). A separate AI agent (GPT-6.1-Sol) carried out an adversarial proof review and found no required mathematical repair. A third AI agent (Grok 4.7) carried out a contribution and literature review. These internal reviews are not peer review. The manuscript was typeset with the help of an AI agent (Claude Opus 5.5), audited by Grok 4.7, and revised with GPT-6-Astra assistance. The mathematics of [7] is due to its authors, and the results of [8, 3, 10, 9] are due to theirs.

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