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Disk rigidity for attained complete polynomial Crouzeix equality

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Abstract

Relative to the pinned complete Crouzeix inequality and its exterior Faber operations, and to Jin's geometric circle criterion, we prove that actual factor-two equality for one finite matrix-valued polynomial forces the numerical range to be a nondegenerate disk in every finite dimension. We retain the Jensen deficit in finite diagonal dual tests. Strong exterior boundary convergence and a triple-variance Fatou argument then make the diagonal compressed resolvents constant, without a rank-one assumption. An ordered constant test and equality in Parseval produce a fixed matrix polynomial channel that is unitary on the boundary. Its rational inverse is holomorphic in the exterior and vanishes at infinity. The determinant therefore realizes the boundary as a full polynomial level set. The numerical-range Hermitian tangent pencil supplies the additional hypothesis of the credited circle criterion. The argument allows coefficient rank greater than one and assumes no initial boundary smoothness.

1 Statement, dependencies, and conventions

For $A \in M_n(\mathbb{C})$ let

$$W(A) = \{v^* A v : v \in \mathbb{C}^n,\ \|v\| = 1\}.$$

This is a compact convex set containing $\operatorname{spec} A$. All operator norms are Euclidean. For $F(z) = \sum_{k=0}^{D} z^k C_k$, $C_k \in M_m(\mathbb{C})$, we use the base-first evaluation

$$F[A] = \sum_{k=0}^{D} A^k \otimes C_k.$$

The Hilbert–Schmidt norm is unnormalized: $\|C\|_{\mathrm{HS}}^2 = \operatorname{tr}(C^* C)$. A bar denotes entrywise conjugation, T denotes ordinary transpose, and $*$ denotes adjoint. Circle integrals use $d\tau(e^{it}) = dt/(2\pi)$; L^2 norms of matrix fields use the Hilbert–Schmidt norm.

Theorem 1.1. *For all finite integers $n, m \geq 1$, every $A \in M_n(\mathbb{C})$ and every finite $M_m(\mathbb{C})$-valued polynomial F, put*

$$M_0 = \max_{z \in W(A)} \|F(z)\|.$$

If $M_0 > 0$ and $\|F[A]\| = 2M_0$, then $W(A)$ is a closed disk of positive radius.

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There is no dimension or degree restriction, initial irreducibility hypothesis, boundary regularity hypothesis, or simple support eigenvalue hypothesis. The hypothesis concerns one attained polynomial witness; it does not concern a possibly nonattained supremum over witnesses. The proof also gives the following structure.

Theorem 1.2. *Normalize $M_0 = 1$. Let B be any unitarily irreducible reducing block of A with $\|F[B]\| = 2$, and let $K = W(B)$. There are fixed subspaces $E, G \subset \mathbb{C}^m$ of the same positive dimension r and isometries $U : \mathbb{C}^r \to E$, $V : \mathbb{C}^r \to G$ such that*

$$J(z) = V^* F(z) U$$

is a square polynomial, $F(z)E \subset G$ for every z, and J is unitary on ∂K. Its inverse $Q = J^{-1}$ is rational, holomorphic on $\mathbb{C} \setminus K$, continuous up to ∂K, and $Q(\infty) = 0$. The nonconstant polynomial $f = \det J$ satisfies the full identities

$$\{|f| < 1\} = \operatorname{int} K, \qquad \{|f| = 1\} = \partial K. \tag{1.1}$$

We assume the complete inequality and analytic exterior Faber operations of OpenAI [1] at the immutable revision

`fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`.

These are stipulated inputs. We state the exact operations needed in Section 2, rather than auditing that upstream theorem. The structural companion [2] fixes the distinction between the original matrix's mass-two positive resolvent and the normalized positive representation of a similar matrix. We use the original field. The only external specialized geometric results used are Jin v2 Lemma 8 (a curved exposed arc) and Proposition 6 (the circle criterion) [3]; their hypotheses are stated at their applications. Jin's scalar equality theorem is not applied to the matrix witness. Classical conformal mapping, Herglotz representation, Jordan boundary extension, Hardy uniqueness, and scalar maximum principles are used in their usual one-variable forms. The outer-limit argument is included.

The additional mechanism is the retained finite-test Jensen deficit, its triple-variance limit, and the fixed polynomial channel and rational inverse extracted from the constant diagonal fields. These steps permit $r > 1$. They make no assertion that the specified F has a constant scalar coefficient channel attaining two.

2 Active block and the analytic exterior framework

Normalize F by M_0. A finite reducing decomposition of A gives unitarily irreducible blocks; evaluation decomposes as their direct sum, up to the fixed tensor permutation. Select a block $B \in M_d(\mathbb{C})$ with $\|F[B]\| = 2$. The credited complete bound and $W(B) \subset W(A)$ give

$$2 \le 2 \max_{W(B)} \|F\| \le 2. \tag{2.1}$$

Thus its maximum is one. A scalar block cannot attain two. If $K = W(B)$ had empty interior, translation and rotation would make all quadratic forms real; polarization would make the transformed B Hermitian, contradicting irreducibility for $d \ge 2$.

If $Bv = \zeta v$ at a boundary point with $\|v\| = 1$, choose a supporting unit normal ν. The positive Hermitian slack $\operatorname{Re}(\bar\nu\zeta) I - \operatorname{Re}(\bar\nu B)$ has zero quadratic form at v and annihilates it. Combining this with $Bv = \zeta v$ gives $B^* v = \bar\zeta v$. The line through v reduces B, a contradiction. Consequently

$$\operatorname{spec} B \subset \operatorname{int} K. \tag{2.2}$$

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Fix unit top singular vectors x, y with $F[B]x = 2y$, $F[B]^*y = 2x$, and write

$$x = \sum_{i=1}^{m} Xe_i \otimes e_i, \quad y = \sum_{i=1}^{m} Ye_i \otimes e_i, \quad \|X\|_{\mathrm{HS}} = \|Y\|_{\mathrm{HS}} = 1. \tag{2.3}$$

These d-by-m matrices are fixed throughout the approximation.

For clarity we record the exact credited analytic-domain operations from [1, Theorem `lem:faber`, Proposition `prop:coefficients`, Theorems `lem:resolvent` and `thm:domain`]. On a regular real-analytic convex outer domain $\Omega \supset K$, its normalized exterior map has a conformal collar and expansion

$$h(\lambda) = c\lambda + c_0 + O(\lambda^{-1}), \quad c > 0, \qquad q = \lambda h'.$$

The Faber polynomials b_k are specified by

$$\frac{q(\lambda)}{h(\lambda) - z} = \sum_{k\geq 0} b_k(z)\lambda^{-k}, \quad b_0 = 1, \quad \deg b_k = k. \tag{2.4}$$

Their boundary decomposition is

$$b_k(h(\mu)) = \mu^k + \sum_{\ell\geq 1} s_{k\ell}\mu^{-\ell} \quad (k \geq 1). \tag{2.5}$$

The mixed series is absolutely uniformly convergent on $\mathbb{T}^2$. The continuous kernel, off its removable diagonal,

$$\kappa(\lambda, \mu) = 2\,\mathrm{Re}\,\frac{q(\lambda)}{h(\lambda) - h(\mu)} \tag{2.6}$$

is nonnegative and has both angular marginals one. Its integral operator

$$(\mathcal{T}u)(\mu) = \int_{\mathbb{T}} \kappa(\lambda, \mu)u(\lambda)d\tau(\lambda)$$

is an L^2 contraction for Hilbert-valued fields. For a finite polynomial $G = \sum_{k=0}^{N} b_k C_k$, put $u(\lambda) = \sum_{k=1}^{N} C_k\lambda^k$. Then

$$G \circ h = C_0 + u + \mathcal{T}u, \tag{2.7}$$

with disjoint constant, positive, and negative Fourier supports. In particular

$$\sum_{k=0}^{N} \|C_k\|_{\mathrm{HS}}^2 \leq \|G \circ h\|_2^2 \leq \|C_0\|_{\mathrm{HS}}^2 + 2\sum_{k=1}^{N} \|C_k\|_{\mathrm{HS}}^2 . \tag{2.8}$$

The original resolvent field is

$$R = q(hI - B)^{-1} = \sum_{k\geq 0} b_k(B)\lambda^{-k}, \qquad H = R + R^* \succeq 0. \tag{2.9}$$

At fixed Ω this series is absolutely uniformly convergent and has mean I. Positivity also follows directly by congruence:

$$(hI - B)^*H(hI - B) = \bar{q}(hI - B) + q(hI - B)^* \succeq 0.$$

Set $P = X^*RX$, $Z = Y^*RY$, $M = Y^*RX$, $L = X^*RY$, and use subscript k for the coefficient of λ^{-k}. Put $a^2 = \sum_{k\geq 0} \|M_k\|_{\mathrm{HS}}^2$. For any finite polynomial $G = \sum b_k C_k$, the base-first tensor convention gives the bilinear pairing

$$y^*G[B]x = \sum_{k,i,\ell} (M_k)_{i\ell}(C_k)_{i\ell}. \tag{2.10}$$

For a normalized polynomial $\widetilde{F}$ with singular value γ and the same pair x, y, the two multiplication orders give

$$y^*(\widetilde{F}G)[B]x = \gamma x^* G[B]x, \qquad y^*(G\widetilde{F})[B]x = \gamma y^* G[B]y. \tag{2.11}$$

Cauchy–Schwarz in (2.10), the lower bound (2.8), and $\left\|\widetilde{F}\right\| \le 1$ on the boundary imply

$$\gamma|x^*G[B]x| \le a\,\|G\circ h\|_2\,, \qquad \gamma|y^*G[B]y| \le a\,\|G\circ h\|_2\,. \tag{2.12}$$

The products are kept in their displayed orders. These formulas and analytic convex outer approximation are the credited Faber input; all equality and limiting deductions below are written out.

3 Dimension-independent exterior convergence

Translate an interior point of K to zero. By [1, Lemma `lem:approx`], take regular analytic convex outer domains converging to K in Hausdorff distance. For fixed $0 < \rho < R_0$ arrange

$$\rho\overline{\mathbb{D}} \subset K \subset \Omega_j \subset R_0\mathbb{D}.$$

Write their maps as $h_j = c_j\lambda + c_{0j} + O(\lambda^{-1})$, $c_j > 0$, and put $q_j = \lambda h_j'$. We include the convergence details since strong convergence, rather than weak convergence, will be used in quadratic expressions.

Lemma 3.1. *After passing to a subsequence, $h_j|_{\mathbb{T}} \to h|_{\mathbb{T}}$ uniformly and $q_j \to q$ strongly in $L^2(\mathbb{T})$, with $|q_j|, |q| \le 4R_0$ and $q \ne 0$ almost everywhere. The map h is the normalized conformal bijection from the exterior disk onto $\mathbb{C}\setminus K$ and extends to a homeomorphism $\mathbb{T} \to \partial K$. Its boundary derivative is q/λ, and q is an outward normal almost everywhere. Moreover*

$$R_j = q_j(h_jI - B)^{-1} \longrightarrow R = q(hI - B)^{-1} \tag{3.1}$$

strongly in L^2, with a common L^∞ bound, and $H_j = R_j + R_j^* \to H = R + R^* \succeq 0$ in L^2.

Proof. Schwarz's lemma applied to $\rho/h_j(1/w)$ in the unit disk gives $c_j \ge \rho$. Applied to $1/\Phi_j(R_0/w)$, where $\Phi_j = h_j^{-1}$, it gives $c_j \le R_0$. The mean of $h_j|_{\mathbb{T}}$ gives $|c_{0j}| \le R_0$. On $\mathbb{T}$, the tangent is $i\lambda h_j'$ and its argument derivative is $\operatorname{Re}(1 + \lambda h_j''/h_j') \ge 0$ by convexity. The harmonic minimum principle and Herglotz representation in the reciprocal disk yield probability measures μ_j on $\mathbb{T}$ with

$$1 + \lambda h_j''/h_j' = \int_{\mathbb{T}} \frac{\lambda + \xi}{\lambda - \xi}\, d\mu_j(\xi),$$
$$\ell_j(\lambda) := \log(h_j'/c_j) = 2\int_{\mathbb{T}} \log(1 - \xi/\lambda)\, d\mu_j(\xi). \tag{3.2}$$

The logarithms are the branches zero at infinity. The sign follows from $\lambda(d/d\lambda)\log(1 - \xi/\lambda) = \xi/(\lambda - \xi)$. Their real parts are at most $\log 4$, so $|h_j'| \le 4R_0$.

Pass to $c_j \to c \ge \rho$ and weak convergence $\mu_j \to \mu$. The coefficient of λ^{-k} in ℓ_j is $-2\int \xi^k d\mu_j/k$. Therefore

$$\sum_{k>N} |\widehat{\ell}_j(-k)|^2 \le 4\sum_{k>N} k^{-2} \tag{3.3}$$

uniformly. Coefficient convergence and this tail bound imply strong L^2 convergence $\ell_j \to \ell$. The limit is finite almost everywhere and $\operatorname{Re}\ell \le \log 4$. For $\operatorname{Re} s, \operatorname{Re} t \le \log 4$,

$$|e^s - e^t| = \left|(s - t)\int_0^1 e^{t+\theta(s-t)}d\theta\right| \le 4|s - t|.$$

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Thus $h_j' = c_j e^{\ell_j} \to c e^{\ell}$ strongly in L^2 and the latter is nonzero almost everywhere. This proves the derivative and q_j convergence with their upper bounds; no lower bound for $|h_j'|$ is used.

The derivative bound makes $h_j|_{\mathbb{T}}$ equi-Lipschitz, and their values are bounded by R_0. Subsequence compactness gives a uniform boundary limit consistent with a locally uniform exterior holomorphic limit. Indeed $|h_j - c_j\lambda| \le 2R_0$ on and outside $\mathbb{T}$ by the maximum principle. The leading coefficient $c \ge \rho$ prevents a constant limit, so the limit is univalent. Every point of K is omitted by h by Hurwitz. For completeness, the inverses Φ_j on compact subsets of $\mathbb{C} \setminus K$ satisfy

$$|\Phi_j(w)| \le (|w| + 2R_0)/\rho.$$

A diagonal normal-family limit Φ has $|\Phi| \ge 1$ and is nonconstant: $|\Phi(w)| \ge (|w| - 2R_0)/R_0$ at large w. The minimum-modulus principle gives $|\Phi| > 1$. Passing in the compositions proves $h \circ \Phi = \mathrm{id}$ and $\Phi \circ h = \mathrm{id}$. This identifies the exterior domain. Uniform boundary convergence and Hausdorff convergence show $h(\mathbb{T}) = \partial K$. Since ∂K is a Jordan curve, the Jordan boundary theorem makes this extension a homeomorphism. In particular boundary injectivity is not inferred from uniform convergence alone.

Local holomorphic convergence identifies the exterior derivative with h', and Hardy coefficient uniqueness identifies its boundary trace with ce^{ℓ}. Equivalently one may integrate on exterior circles and take their L^2 boundary limits. Passing the inequalities $\mathrm{Re}[\bar{q}_j(\lambda)(h_j(\lambda) - w)] \ge 0$ on an almost-everywhere convergent subsequence, first for a countable dense subset of K and then for all $w \in K$, makes q an outward normal almost everywhere.

Finally (2.2) gives a compact neighborhood of ∂K on which the actual fixed resolvent $(zI - B)^{-1}$ is bounded. Uniform convergence of h_j gives uniform convergence of these resolvents; the derivative convergence proves (3.1) and the common boundedness. Positivity of H_j passes on an almost-everywhere subsequence. This uses no resolvent estimate in terms of spectral distance alone. □

4 Equality residuals and constant diagonal fields

Use subscripts j on the fields P_j, Z_j, M_j, L_j defined in Section 2. Put $S = X^*X$, $T = Y^*Y$, $t_j = \max_{\overline{\Omega}_j} \|F\| \to 1$, and $\gamma_j = 2/t_j \to 2$. Apply the source tests to F/t_j with the fixed pair (2.3). All compressed fields converge strongly in L^2 by Lemma 3.1; therefore all their Fourier arrays converge strongly in ℓ^2. Set

$$a_j^2 = \|M_j\|_2^2 \to a^2, \quad D_{Xj} = \left\|P_j + P_j^*\right\|_2^2, \quad D_{Yj} = \left\|Z_j + Z_j^*\right\|_2^2.$$

Lemma 4.1. *There are nonnegative residuals*

$$\begin{aligned} E_{Xj} &= 4a_j^2/\gamma_j^2 - D_{Xj}, & E_{Yj} &= 4a_j^2/\gamma_j^2 - D_{Yj}, \\ E_{Bj} &= D_{Xj} + D_{Yj} - 2\left\|M_j + L_j^*\right\|_2^2, & Z_j^{\mathrm{res}} &= 3\,\|Y^*X\|_{\mathrm{HS}}^2 + \sum_{k\ge 1} \|X^* b_{kj}(B) Y\|_{\mathrm{HS}}^2, \end{aligned} \tag{4.1}$$

satisfying

$$E_{Xj} + E_{Yj} + E_{Bj} + 2Z_j^{\mathrm{res}} = 2(4/\gamma_j^2 - 1)a_j^2 \longrightarrow 0. \tag{4.2}$$

Consequently $Y^*X = 0$, $L = 0$, $a > 0$, and, almost everywhere,

$$\Delta J_0 \Delta = 0, \quad \Delta = \begin{pmatrix} P + P^* & M^* \\ M & Z + Z^* \end{pmatrix} \succeq 0, \quad J_0 = \mathrm{diag}(I_m, -I_m). \tag{4.3}$$

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Proof. The weighted finite dual tests in (2.12) and the upper bound (2.8) give

$$\|S\|_{\mathrm{HS}}^2 + \tfrac{1}{2}\sum_{k\geq 1}\|X^* b_{kj}(B)X\|_{\mathrm{HS}}^2 \leq a_j^2/\gamma_j^2,$$

and the same assertion for Y, T. Parseval gives D_{Xj} as four times the displayed left side, so $E_{Xj}, E_{Yj} \geq 0$. The positive compression $\Delta_j = [X\ Y]^* H_j [X\ Y]$ has lower off-diagonal block $M_j + L_j^*$. Thus

$$0 \leq \operatorname{tr}(\Delta_j J_0 \Delta_j J_0) = \left\|\Delta_j^{1/2} J_0 \Delta_j^{1/2}\right\|_{\mathrm{HS}}^2 \tag{4.4}$$

proves $E_{Bj} \geq 0$ after integration. The constant coefficient of $M_j + L_j^*$ is $2Y^*X$. Its other coefficients are M_{kj} and L_{kj}^* at opposite frequencies. Hence

$$\left\|M_j + L_j^*\right\|_2^2 = a_j^2 + Z_j^{\mathrm{res}}.$$

Adding gives (4.2). The right side tends to zero because a_j is bounded. Every residual therefore tends to zero, including $Y^*X = 0$ and $\|L_j\|_2 \to 0$. Strong convergence gives $L = 0$. Also $a_j^2/\gamma_j^2 \geq \|S\|_{\mathrm{HS}}^2 > 0$, so $a > 0$.

For the quadratic passage, expand

$$\Delta_j J_0 \Delta_j J_0 - \Delta J_0 \Delta J_0 = (\Delta_j - \Delta) J_0 \Delta_j J_0 + \Delta J_0 (\Delta_j - \Delta) J_0.$$

Strong L^2 convergence and trace Cauchy–Schwarz give convergence in L^1. The limiting nonnegative trace has integral zero by $E_{Bj} \to 0$. Thus $\Delta^{1/2} J_0 \Delta^{1/2} = 0$ almost everywhere, and multiplying by $\Delta^{1/2}$ on both sides gives (4.3). No convergence of matrix square roots is needed. □

We now retain the deficit that the non-strict coefficient bound discards. Write $A_{kj} = X^* b_{kj}(B) X$ and let $\mathcal{T}_j$ be the kernel operator (2.6) for h_j.

Lemma 4.2 (Finite diagonal test with its deficit)**.** *For each fixed j and finite N, set*

$$C_0 = \overline{S}, \qquad C_k = \tfrac{1}{2}\overline{A_{kj}}\ (1 \leq k \leq N), \qquad G_{jN} = \sum_{k=0}^{N} b_{kj} C_k,$$

$$u_{jN}(\lambda) = \tfrac{1}{2}\sum_{k=1}^{N} \overline{A_{kj}}\lambda^k, \qquad d_{jN} = \|S\|_{\mathrm{HS}}^2 + \tfrac{1}{2}\sum_{k=1}^{N}\|A_{kj}\|_{\mathrm{HS}}^2, \qquad \delta_{jN} = \|u_{jN}\|_2^2 - \|\mathcal{T}_j u_{jN}\|_2^2.$$

Then $\delta_{jN} \geq 0$ and

$$\|G_{jN} \circ h_j\|_2^2 = d_{jN} - \delta_{jN}, \qquad \gamma_j^2 d_{jN}^2 \leq a_j^2 (d_{jN} - \delta_{jN}). \tag{4.5}$$

At fixed j, let $N \to \infty$. The limits u_j, d_j, δ_j exist in the corresponding L^2 or scalar senses, and

$$0 \leq \delta_j \leq d_j - \gamma_j^2 d_j^2/a_j^2 \longrightarrow 0, \quad d_j = D_{Xj}/4 \to a^2/4, \quad u_j = \tfrac{1}{2}\overline{P_j - S} \to u = \tfrac{1}{2}\overline{P - S} \text{ in } L^2. \tag{4.6}$$

Proof. The three disjoint Fourier pieces (2.7) have squared norm $\|S\|_{\mathrm{HS}}^2 + \|u_{jN}\|_2^2 + \|\mathcal{T}_j u_{jN}\|_2^2 = d_{jN} - \delta_{jN}$. The paired diagonal value $x^* G_{jN}[B]x$ is exactly d_{jN} by the bilinear pairing, and (2.12) proves the second inequality. Contractivity gives nonnegativity of the deficit. At fixed j, square summability of A_{kj} gives $u_{jN} \to u_j$; contractivity gives convergence of its image and of the deficits. Taking $N \to \infty$ yields the displayed bound. Lemma 4.1 gives $D_{Xj} \to a^2$ and $\gamma_j \to 2$, so its right side tends to zero. Strong convergence of P_j gives the last assertion. Only finite polynomials have been inserted in the ordered tests. □

Lemma 4.3 (Hilbert-valued triple variance)**.** *Let $\kappa_j \geq 0$ have both angular marginals one. For a field u_j with values in a finite-dimensional Hilbert space and its integral operator $\mathcal{T}_j$, one has*

$$2(\|u_j\|_2^2 - \|\mathcal{T}_j u_j\|_2^2) = \iiint_{\mathbb{T}^3} \kappa_j(\lambda,\mu)\kappa_j(\nu,\mu)\,\|u_j(\lambda) - u_j(\nu)\|^2\, d\tau(\lambda)d\tau(\nu)d\tau(\mu). \tag{4.7}$$

Proof. Expand the squared difference. For fixed μ, the two squared-norm terms each integrate to $\int \kappa_j(\lambda,\mu)\,\|u_j(\lambda)\|^2\, d\tau(\lambda)$ because the ν marginal is one. The cross term is $-2\,\|\mathcal{T}_j u_j(\mu)\|^2$. Integrating in μ and using the other marginal gives (4.7). Truncation or Tonelli justifies the expansion for L^2 fields. The factor two is retained. □

Proposition 4.4 (Constant diagonal fields at arbitrary rank)**.** *The limiting fields satisfy*

$$X^*R(\lambda)X = S, \qquad Y^*R(\lambda)Y = T \quad \textit{for almost every } \lambda \in \mathbb{T}. \tag{4.8}$$

Proof. Select a common subsequence with $q_j \to q$ and $u_j \to u$ almost everywhere. Uniform convergence of h_j and injectivity of $h|_{\mathbb{T}}$ imply, away from null diagonals and derivative exceptional sets,

$$\kappa_j(\lambda,\mu) \longrightarrow \kappa(\lambda,\mu) = 2\,\mathrm{Re}\,\frac{q(\lambda)}{h(\lambda) - h(\mu)} \geq 0.$$

Apply Fatou to the nonnegative triple integrand in (4.7). By (4.6),

$$\kappa(\lambda,\mu)\kappa(\nu,\mu)\,\|u(\lambda) - u(\nu)\|_{\mathrm{HS}}^2 = 0 \quad \text{for almost every triple.} \tag{4.9}$$

No marginal identity for the limit kernel is asserted. No uniform strict-contraction constant or operator-norm kernel convergence is used.

Jin [3, Lemma 8], applied using (2.2), supplies a nonempty open regular analytic boundary arc of positive curvature with singleton exposed faces. Shrink it so that each point lies on no nontrivial boundary segment. Its preimage under the boundary homeomorphism is a nonempty open circle interval I, with positive Haar measure. For $\mu \in I$ and almost every $\lambda \neq \mu$, the nonzero outward normal $q(\lambda)$ has strictly positive supporting gap against $h(\mu)$. A zero gap would place the two points on the same support line, making their segment a boundary segment containing $h(\mu)$. Therefore $\kappa(\lambda,\mu) > 0$ for these pairs. Fubini permits a choice of $\mu \in I$ for which (4.9) holds for almost every (λ,ν). It follows that u is constant almost everywhere. Its mean is zero, so $u = 0$ and $P = S$. Repeating Lemmas 4.2–4.3 and this argument with the second ordered test, Y, and Z_j gives $Z = T$. □

5 Rational cross field and the fixed coefficient channel

Define rational functions in the physical variable by

$$C(z) = Y^*(zI - B)^{-1}X, \qquad r_0(z) = \mathrm{tr}[X^*(zI - B)^{-1}X]. \tag{5.1}$$

Since $\mathrm{tr}\,S = 1$, (4.8) gives $q\,r_0 \circ h = 1$ almost everywhere. The compressed exterior resolvent is bounded and holomorphic, including at infinity. Hardy uniqueness in the reciprocal disk extends its constant boundary identity throughout the exterior. Hence

$$q(\lambda) = 1/r_0(h(\lambda)), \qquad M(\lambda) = \mathcal{M}(h(\lambda)), \quad \mathcal{M}(z) = C(z)/r_0(z). \tag{5.2}$$

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We check nonvanishing of the denominator in both the open exterior and on the boundary. For $z \notin K$, strict separation supplies a unit ν such that $D = \bar{\nu}(zI - B)$ has positive definite real part. Then

$$\operatorname{Re}(D^{-1}) = D^{-*}(\operatorname{Re} D)D^{-1} \succ 0, \qquad D^{-1} = \nu(zI - B)^{-1}.$$

As XX^* is nonzero and positive,

$$\operatorname{Re}[\nu r_0(z)] = \operatorname{tr}(XX^* \operatorname{Re} D^{-1}) > 0. \tag{5.3}$$

Thus there is no exterior zero. By spectral separation r_0 is continuous near ∂K. The boundary identity and $|q| \le 4R_0$ give $|r_0 \circ h| \ge (4R_0)^{-1}$ almost everywhere on $\mathbb{T}$. This continuous function satisfies the same inequality everywhere: failure at one point would give failure on an open circle interval, of positive Haar measure. The boundary homeomorphism transfers the bound to all of ∂K. This uses full angular support, without assuming an unproved equivalence with arclength measure.

It follows that $\mathcal{M}$ is rational, holomorphic outside K and in a neighborhood of its boundary, and continuous up to the boundary. Since $Y^*X = 0$, $C(z) = O(z^{-2})$ and $r_0(z) = z^{-1} + O(z^{-2})$, so

$$\mathcal{M}(z) = O(z^{-1}) \quad (z \to \infty). \tag{5.4}$$

Now (4.3) and (4.8) give

$$\Delta = \begin{pmatrix} 2S & M^* \\ M & 2T \end{pmatrix}, \qquad M^*M = 4S^2, \quad MM^* = 4T^2, \quad MS = TM. \tag{5.5}$$

In particular S, T have the same positive eigenvalues and equal positive rank r; $\ker M = \ker S$ and $\operatorname{ran} M = \operatorname{ran} T$ almost everywhere.

Lemma 5.1 (The ordered constant test)**.** *On the boundary, almost everywhere,*

$$F(h)\overline{S} = \tfrac{1}{2}\overline{M}. \tag{5.6}$$

Proof. Put $s = \|S\|_{\mathrm{HS}} > 0$. Constancy of P gives $D_{Xj} \to 4s^2$, so the vanished residual yields $a = 2s$. Use the constant polynomial $G = \overline{S}$ in the first identity (2.11). The Faber coefficients of $(F/t_j)\overline{S}$ are finite, of degree at most $D = \deg F$. They converge to the nonnegative Fourier coefficients U_k of $F(h)\overline{S}$: this follows from uniform boundary convergence, since the nonnegative part of each Faber polynomial is exactly its leading circle power. Thus $U_k = 0$ for $k > D$. Passing the finite pairing (2.10) gives the positive real identity

$$2s^2 = \sum_{k \ge 0, i, \ell} (M_k)_{i\ell}(U_k)_{i\ell}.$$

Cauchy–Schwarz, Parseval, and $\|F(h)\| \le 1$ give

$$2s^2 \le a\Big(\sum_{k \ge 0} \|U_k\|_{\mathrm{HS}}^2\Big)^{1/2} \le a \left\|F(h)\overline{S}\right\|_2 \le as = 2s^2. \tag{5.7}$$

Every step is equality. Equality in the bilinear coefficient Cauchy–Schwarz inequality, with the positive real phase, gives $U_k = \overline{M_k}/2$ for every k, including $M_k = 0$ above degree D. Equality in the Parseval step removes the entire negative Fourier part of $F(h)\overline{S}$. Finally $M_0 = Y^*X = 0$. Combining the full Fourier expansions proves (5.6). The order $F\overline{S}$, entrywise conjugation, and factor $1/2$ follow from the stated pairing; none can be replaced by an adjoint or a reversed product. □

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Take $E = \operatorname{ran}\overline{S}$, $G = \operatorname{ran}\overline{T}$ and fixed isometries $U : \mathbb{C}^r \to E$, $V : \mathbb{C}^r \to G$. Equations (5.5)–(5.6) give

$$(F(h)\overline{S})^*(F(h)\overline{S}) = \tfrac{1}{4}M^{\mathsf{T}}\overline{M} = \overline{S}^{\,2}.$$

Thus $F(h)$ is isometric on E and maps it into G almost everywhere. The polynomial $(I - VV^*)F(z)U$ vanishes at infinitely many distinct boundary points, hence vanishes identically. Therefore $J = V^*FU$ is a square polynomial, unitary almost everywhere on the boundary, and everywhere there by continuity and full angular support.

Set $S_E = U^*\overline{S}U \succ 0$. The restriction calculation is

$$\begin{aligned} \overline{S}U &= US_E, \\ J(h)S_E &= V^*F(h)\overline{S}U = \tfrac{1}{2}V^*\overline{M}U, \\ S_E J(h)^* &= \tfrac{1}{2}U^*M^{\mathsf{T}}V. \end{aligned} \tag{5.8}$$

Consequently the rational matrix

$$Q(z) = \tfrac{1}{2}S_E^{-1}U^*\mathcal{M}(z)^{\mathsf{T}}V \tag{5.9}$$

is holomorphic outside K, continuous at its boundary, and $Q(\infty) = 0$. The transpose in (5.9) is ordinary transpose, not adjoint. The inverse Gram factor multiplies on the left. On the boundary $Q = J^*$, so $JQ = I$. Uniqueness of rational identities on infinitely many boundary points gives

$$J(z)Q(z) = I \tag{5.10}$$

identically as a rational identity. In particular $Q = J^{-1}$ throughout the exterior. This proves the fixed-channel and inverse assertions of Theorem 1.2.

6 Full determinant level, geometry, and restoration

For unit vectors u, v, apply the exterior scalar maximum principle to u^*Qv. It is bounded by one on the boundary and zero at infinity. Thus $\|Q\| \le 1$ outside K. If equality held at an exterior point, choose norming u, v there; the scalar maximum principle would make u^*Qv constant of modulus one, contradicting its value at infinity. Therefore

$$\|Q(z)\| < 1 \quad (z \notin K). \tag{6.1}$$

Let $f = \det J$. This is a nonconstant polynomial: from (5.10), $\det J \det Q = 1$, whereas $Q(\infty) = 0$ implies $\det Q \to 0$. In the interior, the matrix maximum principle gives $\|J\| \le 1$, and hence $|f| \le 1$; nonconstancy and the scalar maximum principle make this strict. On the boundary $|f| = 1$. Outside, (6.1) means every singular value of J is greater than one, so $|f| > 1$. We have proved the full identities

$$\{z \in \mathbb{C} : |f(z)| < 1\} = \operatorname{int} K, \qquad \{z \in \mathbb{C} : |f(z)| = 1\} = \partial K. \tag{6.2}$$

This proves the remaining assertion of Theorem 1.2. It explicitly excludes any additional level components in the exterior.

We verify regularity before applying the geometric input. If $f'(z_0) = 0$ at a boundary point, write $f(z) - f(z_0) = (z - z_0)^k g(z)$ with $k \ge 2$, $g(z_0) \ne 0$. Choose nonzero α so $\operatorname{Re}(\overline{f(z_0)}\alpha^k g(z_0)) < 0$. For all sufficiently small $t > 0$, the k points $z_0 + t\alpha e^{2\pi i\ell/k}$, $0 \le \ell < k$, have $|f| < 1$. By (6.2) they lie in $\operatorname{int} K$. Their average is z_0, contradicting convexity of the interior. Hence $f' \ne 0$ on all of ∂K, which is smooth real analytic by the implicit function theorem. If it contained a nontrivial

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segment, the real polynomial $|f(z_0+t\xi)|^2-1$ would vanish on an interval and hence on the entire real line. This contradicts compactness of the full level (6.2). Thus K is strictly convex.

Here is the exact credited circle criterion [3, Proposition 6]: if a compact strictly convex body has boundary

$$\partial K=\{z: V_0(z)\neq 0,\ |U_0(z)/V_0(z)|=1\},$$

for coprime polynomials with $\deg U_0>\deg V_0$, with $(U_0/V_0)'\neq 0$ on that boundary, and if the tangents to an open boundary arc lie in a Hermitian determinant curve $\det(sI+uH_0+vL_0)=0$ in line coordinates $s+uX+vY=0$, then the boundary is a circle. Its projective-dual contact argument, including Jin's Proposition 5, is credited geometry, not an additional theorem claimed here.

Apply it with $U_0=f$, $V_0=1$. All its level, degree, regularity, and strict-convexity hypotheses have just been checked. For the remaining hypothesis take

$$H_0=\operatorname{Re}B,\qquad L_0=\operatorname{Im}B.$$

Every tangent to the smooth convex boundary is a support line. Orient $s+uX+vY=0$ so $sI+uH_0+vL_0\succeq 0$. A unit vector attaining a point on that line has zero quadratic form for this pencil and lies in its kernel. Consequently every such tangent satisfies

$$\det(sI+u\operatorname{Re}B+v\operatorname{Im}B)=0. \tag{6.3}$$

This pays the Hermitian determinant hypothesis. A convex polynomial lemniscate alone would not pay it. The credited criterion now gives $K=a_0+\rho_0\overline{\mathbb{D}}$ with $\rho_0>0$.

It remains to recover all reducing summands, not just the active block. The polynomial $g(z)=f(a_0+\rho_0 z)$ has modulus one on $\mathbb{T}$. Its Laurent identity

$$g(z)\overline{g(1/\bar z)}=1\quad(z\neq 0)$$

follows from equality on the circle. It forbids any nonzero root. The fundamental theorem of algebra therefore gives $g(z)=\eta z^k$, $|\eta|=1$, $k\geq 1$. For every $z\in W(A)$,

$$\|J(z)\|\leq\|F(z)\|\leq 1,\qquad |f(z)|\leq 1.$$

The monomial identity forces $W(A)\subset a_0+\rho_0\overline{\mathbb{D}}$; $W(B)\subset W(A)$ gives the reverse inclusion. Restoring the initial normalization proves Theorem 1.1.

7 Scope, antecedents, and verification

The result concerns all finite matrix and coefficient dimensions, but only actual polynomial equality with positive maximum. It gives no uniform strict gap on nondisks, stability estimate, rational or arbitrary holomorphic witness extension, nonattained-supremum conclusion, or classification of all equality witnesses.

Jin's scalar equality theorem is all-dimensional, while the complete constant comparison of Åhag–Czyż–Virtanen [4, Theorem 1.6 and Corollary 12.4] gives complete disk rigidity through base dimension three. Their scalar polynomial classification in Theorem 12.2 is also credited prior work. These statements do not provide the arbitrary-rank channel extracted here.

Earlier project result R016 proves specified-witness scalarization and disk rigidity through base dimension four. The campaign's dimension-five extension is a credited technical antecedent. Their

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dimension-independent outer convergence and residual mechanism are reproduced in Sections 3–4; their later rank-one argument is not used. Earlier result R019 gives an attained rank-two matrix polynomial witness on a six-dimensional disk matrix for which no constant scalar coefficient compression attains two. It remains compatible with the present theorem. These antecedents are not new results of this paper. Their exact project records and hashes are listed in the accompanying provenance and source map.

The contribution asserted for review is the deficit-to-constant-field bridge and its fixed matrix-channel determinant continuation, relative to the credited inputs. This is not a worldwide priority assertion. Independent proof and contribution reviews of the frozen discovery argument have respectively returned PASS and SUBSTANTIAL, with bounded primary-literature coverage. Those verdicts do not replace review of this presentation or establish worldwide priority. This is a local manuscript prepared for review; no public submission is represented as having occurred.

The associated discovery checker has 17 exact finite algebra checks in $\mathbb{Q}(\sqrt{2}, \sqrt{7}, i)$. An independent reviewer supplied 16 additional checks over $\mathbb{Q}(i)$, including rectangular non-real isometries and a non-scalar positive Gram matrix. These exercise the factor $1/2$, transpose, multiplication order, and variance factor, with rejected deliberate mutations. They support finite identities; they do not prove the exterior limit, Fatou argument, geometric input, or full theorem. Reproduction commands and their unchanged source hashes are recorded alongside this paper.

No full Lean certificate exists for this theorem. Required analytic formalization would include exterior conformal boundary convergence, Hilbert-valued Markov variance and its Fatou passage, Hardy uniqueness, resolvent separation, matrix maximum principles, and the credited circle criterion. Formalizing only the finite Gram algebra would not certify the analytical theorem.

Computational assistance. OpenAI Codex assisted in developing and checking the written argument, preparing this manuscript, and running supporting exact arithmetic and typesetting checks. The package records the scope of independent written review and the finite computations. Computational output is not used as a substitute for the proof above, and no additional human author or affiliation is implied.

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