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Uniform interior gradient estimates for finite p through a constrained affine lift

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Abstract

Assuming the interior regularity and fitted-class Liouville results of the pinned OpenAI infinity-harmonic manuscript, we give an argument for a dimension-dependent interior $C^{1,\alpha}$ estimate common to all finite $p \geq 3$. The argument uses the exact equation for the affine lift $u(y)+b$, including at critical points of u. Its anisotropic limits have an effective viscosity. Finite viscosity is handled by fitted-class rigidity, whose positive-viscosity proof is included. Infinite viscosity is handled by a uniform small-solution estimate and a quadratic comparison between different fitting scales. The exponent is nonexplicit. Supporting Lean files certify the zero-jet vector ellipticity algebra, not the analytic theorem.

1 Statement and inputs

For a symmetric matrix X and vector v, write $X[v] = v^T X v$. An operator with positive Hessian sign is nonnegative at upper tests of a viscosity subsolution and nonpositive at lower tests of a supersolution. Weak p-harmonic functions are identified with their continuous representatives.

Theorem 1.1. *For every integer $n \geq 2$ there are $\alpha_n \in (0, 1/3]$ and $C_n < \infty$ such that for every finite $p \geq 3$ and every bounded weak solution $u \in W^{1,p}_{\mathrm{loc}}(B_1 \subset \mathbb{R}^n)$ of*

$$\int |Du|^{p-2} Du \cdot D\phi = 0 \qquad (\phi \in C_c^\infty(B_1)),$$

one has $u \in C^{1,\alpha_n}(B_{1/2})$ and

$$\|Du\|_{L^\infty(B_{1/2})} + [Du]_{C^{0,\alpha_n}(B_{1/2})} \leq C_n \operatorname{osc}_{B_1} u. \tag{1}$$

There is no hypothesis on sign, critical points, level sets or boundary data.

The theorem is relative to the stated results of [1] at commit

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Specifically, we use its infinity-harmonic interior estimate in dimensions at least three, its exact infinity-harmonic linear lift, its zero-viscosity fitted-class Liouville theorem, and its equation-free geometric fit and envelope lemmas. The new PDE identifications are proved here. We do not assume that an affine lift of a finite-p solution is p-harmonic. The inhomogeneous equation and the range $2 < p < 3$ are outside the statement.

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The classical analytic inputs are the local logarithmic gradient estimate of Kotschwar–Ni [2], scalar regularity from Uhlenbeck [3], weak/viscosity equivalence [5], Savin's small-solution theorem [4], and linear weak-solution Harnack [6]. The latter is used in Appendix A. Their conclusions are not new claims of this manuscript. A companion audit records exact source versions, read scopes and the compact-parameter choices in Uhlenbeck's proof.

Appendix A supplies the positive-viscosity argument: the cone infimum, viscosity-to-weak passage, contact propagation and rigidity. No estimate uniform in the positive viscosity is used.

Relation to existing estimates. Peng–Zhang–Zhou [8, Theorem 1.1] establish C^1 convergence for potentials on convex rings, with geometric and nonvanishing-gradient structure. Bungert [9, Theorem 1] gives value-convergence rates under a value-Hölder hypothesis. Neither statement supplies a common gradient Hölder estimate for arbitrary interior weak solutions. The qualitative-to-quantitative theorem of Sobral–Teixeira–Urbano [10, Theorem 1.1] gives constants depending on p. Its large-shift compactness at fixed p produces a linear operator with matrix $I+(p-2)e\otimes e$, whose ellipticity ratio diverges in the joint large-p limit. The effective-viscosity analysis below addresses this simultaneous degeneracy.

Lee–Lee [11] prove improved homogeneous regularity and the inhomogeneous $C^{p'}$ conclusion for p sufficiently close to two. Their Theorem A transfers a homogeneous exponent strictly larger than $1/(p-1)$ to the sharp bounded-forcing exponent. If Theorem 1.1 is established, that transfer applies for each fixed $p>\max\{3,1+1/\alpha_n\}$. It is a corollary of the present claim, with constants allowed to depend on p; the intervening range is not settled. In particular the full inhomogeneous conjecture is not claimed here.

2 Uniform Lipschitz bounds and qualitative flatness

For a positive p-harmonic function v on a Euclidean ball B_R, put $w=-(p-1)\log v$. In [2, Theorem 1.1], choose the auxiliary parameter $1/2$ and zero curvature bound. For $p\geq 3$ the displayed constants reduce to

$$c_{p,n}=(p/2+1)(p-1),\qquad b_{p,n}=2(p-1)/(n-1)-2,$$
$$C(n,0,p,R,1/2)=\frac{20(n+p-2)(n-1)+10(p-1)(n-1)}{R^2}.$$

Substitution gives $|Dw|\leq K_n(p-1)/R$ on $B_{R/2}$, hence $|D\log v|\leq K_n/R$. If $M=\operatorname{osc} u>0$, replace u by $v=u-\inf u+M$, so $M\leq v\leq 2M$ and $|Du|\leq 2K_nM/R$. The case $M=0$ is immediate. Classical scalar regularity gives the differentiable representative; the gradient estimate's smooth noncritical argument extends across critical points where the gradient is zero.

Normalize $\operatorname{osc}_{B_1} u\leq 1$ and $u(0)=0$. The affine lifts

$$U(y,b)=u(y)+b,\qquad \xi=e_b,$$

are uniformly bounded and L-Lipschitz on $B_{7/8}\subset\mathbb{R}^{n+1}$, with L depending only on n, and satisfy $U(z+h\xi)=U(z)+h$.

Lemma 2.1 (Compact-p interface)**.** *For every $n\geq 2$ and finite $P\geq 3$, there are common $\theta(n,P)>0$ and $C(n,P)<\infty$ such that every bounded weak p-harmonic solution on B_1, $3\leq p\leq P$, satisfies*

$$\|Du\|_{L^\infty(B_{1/2})}+[Du]_{C^{0,\theta}(B_{1/2})}\leq C(n,P)\operatorname{osc}_{B_1} u.$$

This includes zero gradients. An entire weak solution for one fixed $p\geq 3$ with bounded gradient is affine.

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Proof. We give the common choices in [3, Sections 1–5]; a fixed-p estimate alone would not establish uniformity. Work first in dimension $m \geq 3$, and distinguish Uhlenbeck's growth exponent $s = (p-2)/2 \in [1/2, S]$, $S = (P-2)/2$, from the present p. Set $\omega = Du$, $Q = |\omega|^2$, $\rho(Q) = Q^s$, and set her structural constant C to zero. The gradient/curl complex satisfies her (1.2), and ω solves her system (1.1).

Common structural and preliminary estimates. Fix

$$a = \min\{1/2, 1/(m-2)\}, \qquad K_* = 2 + (1+2S) + S\max\{1, S\}.$$

The exponent a is positive, at most every s, and strictly below $2/(m-2)$, so $2(1+a)$ is one fixed admissible Sobolev exponent. The ellipticity expression is $\rho + 2Q\rho' = (1+2s)Q^s$, and

$$|sx^s - sy^s| \leq S\max\{1, S\}(x+y)^{s-a}|x-y|^a \qquad (x, y \geq 0). \tag{2}$$

Indeed divide by $(x+y)^s$ when $x + y > 0$. On $[0, 1]$, the power t^s is s-Hölder with constant one for $s \leq 1$, and is S-Lipschitz for $s \geq 1$. Since $a \leq s$ and $a \leq 1$, either bound implies (2); the zero case is direct. Thus (1.3), (1.3)′ and (1.4) of the source hold with K_*. Differentiability of ρ is required only at $Q > 0$.

The source's (1.7), (1.10) and Sections 2–3 apply to

$$H_s(Q) = \frac{s+1/2}{s+1}Q^{s+1} = \frac{p-1}{p}|Du|^p.$$

Its prefactor is in $[2/3, 1]$. The divergence-form operator in (1.10) has common measurable ellipticity bounds controlled by K_*. Fix a mean-value exponent β strictly between 1 and $m/(m-2)$ in (2.1), and fix the Sobolev exponent. Moreover,

$$|D\sqrt{H_s(Q)}| \leq (S+1)Q^{s/2}|D\omega|.$$

The weighted differentiability and energy estimate (3.1), followed by these fixed Sobolev and Moser estimates in the proof of (3.2), give one $C_2(m, P)$ such that

$$\sup_{B_r} H_s(Q) \leq C_2(m, P)r^{-m}\int_{B_{4r}} Q^{s+1}. \tag{3}$$

The constants here and in (3.3) involve K_*, $(S+1)^2$, fixed cutoff factors, Sobolev and Poincaré constants, and Moser's $c(\beta)$ and weak-Harnack constant $c'(1)$. None requires a parameter-dependent threshold choice. In the scalar case the potential for ω is u itself; no varying-exponent Hodge potential estimate is implicit. The weak differentiability and subsolution justification in Sections 2–3 includes $Q = 0$.

One common small-excess iteration. Fix the relative reference size $\eta = 1/2$ in the source's (4.1). Normalize $\sup Q = M = 1$ by homogeneity; $M = 0$ is trivial. During the iteration maintain $|w_i| \leq 1$ and $|w_i|^2 \geq 1/4$. The normalized constant-coefficient linearization at a reference w is

$$I + 2s\frac{w}{|w|} \otimes \frac{w}{|w|},$$

with eigenvalues in $[1, P-1]$. Its comparison energy and interior estimates in (4.2) therefore have common constants.

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Here is a uniform derivation of the nonlinear error estimate, without leaving the power-map constant of printed (4.6) implicit. Put

$$A_s(v) = |v|^{2s}v, \qquad F_s(v) = |v|^s v, \qquad G_s(v,w) = \frac{A_s(v) - A_s(w) - DA_s(w)(v-w)}{\rho(|w|^2)}.$$

For $q = 2s \in [1, 2S]$, the second derivative of A_s away from zero is bounded by $C_m q(q+3)|v|^{q-1}$. Integration on a segment, split at zero when necessary, shows that DA_s is uniformly Lipschitz on the unit ball. Since $\rho(|w|^2) \geq 4^{-S}$,

$$|G_s(v,w)| \leq C(m,P)|v-w|^{1+a} \quad (|v|, |w| \leq 1,\ |w|^2 \geq 1/4). \tag{4}$$

To invert the power map uniformly, let $R = \max\{|v|, |w|\}$. The inequality $DF_s(z) \geq |z|^s I$, integrated over the quarter of the joining segment nearest a maximal endpoint, gives

$$(F_s(v) - F_s(w)) \cdot (v-w) \geq 2^{-S-2} R^s |v-w|^2.$$

Here that segment has norm at least $R/2$, while $R \geq 1/2$. Cauchy–Schwarz consequently yields

$$|v-w| \leq 2^{2S+2}|F_s(v) - F_s(w)|. \tag{5}$$

Let $E = \int |\omega - w|^2$ on a fixed outer ball. The weighted estimate (4.8) has a common scalar constant: $DA_s(\omega)$ has eigenvalues between Q^s and $(P-1)Q^s$, so testing the differentiated equations against cutoff squared times $\omega_j - w_j$ gives a weighted-energy factor at most $4(P-1)^2$ times fixed cutoff constants. The source's difference-quotient argument justifies this at weak solutions and zeros of ω. Also

$$\|DF_s(v)\| \leq (S+1)|v|^s, \qquad |F_s(v) - F_s(w)| \leq (S+1)|v-w| \quad (|v|, |w| \leq 1).$$

Thus on nested fixed balls,

$$\|F_s(\omega) - F_s(w)\|^2_{W^{1,2}} \leq C(m,P)E.$$

The fixed embedding into $L^{2(1+a)}$, together with (4)–(5) and the linear comparison tested against its zero-boundary difference, shows that the squared L^2 error of the linearized comparison field is at most $C(m,P)E^{1+a}$, as required in (4.3)–(4.4).

Specify the next reference as follows. If $\widetilde{\omega}_i$ is the linearized comparison to $\omega^i - w_i$ on the fixed inner ball, take

$$\widehat{w}_{i+1} = w_i + \widetilde{\omega}_i(0), \qquad w_{i+1} = \mathrm{proj}_{\overline{B}_1}(\widehat{w}_{i+1}).$$

The center value is controlled by the common interior estimate; its gradient bound gives the $O(t^2)$ excess improvement. Projection does not increase the error to any $\omega^i(x) \in \overline{B}_1$ or the distance to $w_i \in \overline{B}_1$. With the fixed half-ball rescalings in (4.9)–(4.11) absorbed into one $C_3 = C_3(m,P) \geq 1$, let E_i be the L^2 excess on the rescaled unit ball and ω^i the corresponding rescaled field. We obtain

$$E_{i+1} \leq C_3(t^2 + t^{-m}E_i^a)E_i, \qquad |w_{i+1} - w_i| \leq C_3\sqrt{E_i}. \tag{6}$$

Choose once

$$\begin{aligned} t &= \min\{1/8, 1/(4C_3)\}, \\ \epsilon &= \min\left\{1, \frac{1}{256C_3^2}, \left(\frac{t^{m+1}}{4C_3}\right)^{1/a}\right\}. \end{aligned} \tag{7}$$

If $E_0 \le \epsilon$, then $E_{i+1} \le (t/2)E_i \le tE_i$, and

$$\sum_i |w_{i+1} - w_i| \le \frac{C_3\sqrt{\epsilon}}{1-\sqrt{t}} \le 2C_3\sqrt{\epsilon} \le 1/8.$$

All references have norm at most one, so their squared norms differ from $|w_0|^2 \ge 1/2$ by at most twice this drift. Hence $|w_i|^2 \ge 1/4$, closing the induction. The source's (4.12)–(4.15), with their common geometric-series ratio, now give (4.1) with one threshold $\epsilon_* > 0$ and one 1/2-Hölder constant $K_4(m, P)$. Fixed unit-ball versus B_2/B_1 conventions only decrease ϵ_* by fixed radius and volume factors.

One common decrease factor, including critical points. Proposition (5.1) uses (3.3), the fixed weak-Harnack constant $c'(1)$, Poincaré for $F_s(\omega)$, and the factors $(s+1)^2$ and 2^{2s+2}. These are bounded by $(S+1)^2$ and 2^{2S+2}; all other factors are the common K_*, cutoff, Sobolev and Moser constants just listed. The radial maps selecting the reference have exponent $s+1 \in [3/2, S+1]$, so neither their exponents nor their reciprocals introduce a degenerating choice. Choose one $C_5(m, P) \ge 1$ for that proposition. The two required inequalities on printed p. 238 are

$$1 - C_5\sqrt{\lambda} \ge 2^{-(s+1)/2}, \qquad \lambda C_5 2^{s+1} \le \epsilon_*.$$

They hold simultaneously for all s with

$$\lambda_* = \min\left\{1/4, \frac{1}{16C_5^2}, \frac{\epsilon_*}{2C_5 2^{S+1}}\right\} > 0.$$

Indeed the first left side is at least $3/4$, while its right side is at most $2^{-3/4} < 3/4$; the second inequality is direct. Set

$$\theta = -\frac{\log(1-\lambda_*)}{2\log 4} > 0, \qquad \sqrt{1-\lambda_*} = 4^{-\theta}.$$

Then $\theta < 1/2$. Repeated maximum decrease in (5.4)–(5.5) gives the r^θ modulus; at the first scale where the perturbation alternative applies, the common 1/2-Hölder bound is stronger. If the decrease continues forever, the maximum tends to zero, covering a critical gradient without linearizing at zero. Consequently

$$[\omega]_{C^{0,\theta}(B_{r/2})} \le K_0(m, P) r^{-\theta}\sqrt{M} \quad \text{if } \sup_{B_r} |\omega|^2 \le M. \tag{8}$$

Translation supplies the estimate at every center.

The oscillation bound and dimension two. Normalize $0 \le u \le 1$ by subtracting its essential infimum and dividing by its essential oscillation; zero oscillation is trivial. A cutoff $\chi = 1$ on $B_{7/8}$, supported in B_1, with $|D\chi| \le 16$, is used in the admissible weak test $\chi^p u$. Hölder's inequality, with the zero-energy case separated, gives

$$\int \chi^p |Du|^p \le p\left(\int \chi^p |Du|^p\right)^{(p-1)/p}\left(\int |u|^p |D\chi|^p\right)^{1/p}.$$

Sobolev approximation justifies the test. Therefore

$$\int_{B_{7/8}} |Du|^p \le (16P)^P |B_1| =: E_*.$$

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Apply (3) with centers in $B_{3/4}$ and $r = 1/32$, so B_{4r} stays in $B_{7/8}$. It gives

$$\sup_{B_{3/4}} |Du|^p \le (3/2)C_2 32^m E_* =: A_*, \qquad \|Du\|_{L^\infty(B_{3/4})} \le \max\{1, A_*\}^{1/3} =: L_*.$$

Use (8) on $B(x, 1/4)$ for $x \in B_{1/2}$. Pairs separated by at most $1/8$ obey that local bound; more distant pairs obey $|Du(x) - Du(y)| \le 2\,8^\theta L_* |x - y|^\theta$. This proves the claimed gradient norm and full seminorm after restoring oscillation. For original dimension two, extend u constantly in a third coordinate. Fubini shows it is an actual weak p-harmonic function on $B_1^2 \times \mathbb{R}$, with the same gradient norm and oscillation. Apply the three-dimensional estimate on B_1^3 and restrict back. This constant extension is distinct from the affine lift $u + b$. Finally, for a single fixed p, scale (8) on arbitrarily large balls for an entire bounded-gradient solution. Its gradient oscillation on any fixed ball tends to zero, so it is affine, also as in the source's Bernstein consequence (6.1). □

The normalized family has a common $o(r)$ Euclidean affine-fit error at interior points. Otherwise choose $p_i \ge 3$, $y_i \in B_{3/4}$, $r_i \downarrow 0$ and rescale

$$f_i(Y) = \frac{u_i(y_i + r_i Y) - u_i(y_i)}{r_i}$$

so they remain a fixed positive distance from every linear function on B_2. The common Lipschitz bound gives a locally uniform entire limit f, with $f(0) = 0$. After a subsequence $p_i \to p_* < \infty$ or $p_i \to \infty$. Weak/viscosity equivalence identifies the continuous polynomial equation

$$D^2 u[Du] + \epsilon |Du|^2 \Delta u = 0, \qquad \epsilon = (p-2)^{-1}. \tag{9}$$

At zero test gradient both this expression and the continuous p-Laplacian vanish. Viscosity stability therefore gives a weak p_*-harmonic limit in the finite case, affine by Lemma 2.1, or an infinity-harmonic limit otherwise. In the latter case $F(Y, b) = f(Y) + b$ is infinity-harmonic in dimension $n + 1 \ge 3$ by the pinned exact lift. Scaling its interior estimate on B_R gives $[DF]_{\alpha; B_{R/2}} \le C \operatorname{Lip}(F) R^{-\alpha}$. Letting $R \to \infty$ makes F, hence f, affine. Both alternatives contradict the chosen error. Adding back the exact b coordinate gives the same qualitative flatness for U.

3 The exact lifted equation

Put $\Pi = I - \xi\xi^T$. The lift satisfies, at all viscosity tests,

$$\mathcal{L}(DU, H) = H[DU] + \epsilon |\Pi DU|^2 \operatorname{tr}(\Pi H) = 0. \tag{10}$$

Here is a proof covering critical points. At an upper test ϕ at the origin, restriction to the b line gives $\phi_b = 1$ and $H_{bb} \ge 0$. For $\sigma > 0$ replace ϕ by $\phi_\sigma = \phi + \sigma b^2/2$. The implicit equation $(\phi_\sigma)_b = 1$ defines a local C^1 branch $b = b(y)$. The function $\psi(y) = \phi_\sigma(y, b(y)) - b(y)$ is a C^2 upper test of u, with gradient $g = D_y \phi$ and Hessian

$$K = H_{yy} - \frac{H_{yb} H_{by}}{H_{bb} + \sigma}.$$

If $H_\sigma = H + \sigma \xi\xi^T$, then $H_\sigma - \operatorname{diag}(K, 0)$ is positive semidefinite, being the outer product of $(H_{yb}, H_{bb} + \sigma)$ divided by $H_{bb} + \sigma$. The matrix $M = (g, 1)(g, 1)^T + \epsilon |g|^2 \Pi$ is positive semidefinite, and

$$\operatorname{tr}(M H_\sigma) \ge \operatorname{tr}((gg^T + \epsilon |g|^2 I) K) \ge 0.$$

Let $\sigma \downarrow 0$; no convergence of K is needed. When $g = 0$, the right-hand side is zero and the vertical curvature is sufficient. Lower tests use $\phi - \sigma b^2/2$ and the negative semidefinite block comparison. This proves (10).

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4 Cylinder selection and compactness

Write $N = n + 1$, $G = B_{3/4} \subset \mathbb{R}^N$ and $D(z) = \operatorname{dist}(z, \partial G)$. For nonzero P put $N_P = P/|P|$ and

$$Z_\lambda(z, r; P) = \{z + h : |h \cdot N_P| \le r,\ |h - (h \cdot N_P)N_P| \le \sqrt{\lambda} r\}.$$

A parameter $\lambda \in (0, 1)$ is good at (U, z, r) if some nonzero P satisfies

$$|U(z + h) - U(z) - P \cdot h| \le \lambda r \qquad (z + h \in Z_\lambda(z, r; P)). \tag{11}$$

Let $\beta(U, z, r)$ be the supremum of bad parameters, with supremum zero for the empty set, and let $b(s)$ be the supremum of these quantities over the normalized family, $z \in G$, and $0 < r \le sD(z)$. We only use $s \le 1/8$, so all cylinders lie in the solution domain. The inscribed ball and exact affine increment imply, for every fit,

$$|P| \le L + \sqrt{\lambda}, \qquad |P \cdot \xi - 1| \le \sqrt{\lambda}. \tag{12}$$

Thus sufficiently accurate fits have $1/2 \le |P| \le L + 1$. The qualitative flatness above implies $b(s) \to 0$: otherwise a bad parameter bounded below persists at radii tending to zero, whereas an affine limit with added-coordinate slope one makes that parameter good. This argument does not require monotonicity of goodness.

Lemma 4.1 (Strengthened scale selection)**.** *If b has no positive power bound near zero, there are $\gamma_i \downarrow 0$, $s_i \to 0$, actual bad parameters λ_i, and associated (U_i, z_i, r_i) with*

$$\frac{b(s)}{s^{\gamma_i}} \le 2\frac{b(s_i)}{s_i^{\gamma_i}} \quad (s_i \le s \le 1/8), \qquad \frac{b(s_i)}{s_i^{\gamma_i}} > i, \tag{13}$$

$$b(s_i)/4 < \lambda_i \le b(s_i), \qquad r_i \le s_i D(z_i). \tag{14}$$

For every fixed positive integer k, these satisfy

$$s_i/\lambda_i^k \longrightarrow 0. \tag{15}$$

Proof. Take $\gamma_i = 1/i$. Failure of every power bound makes $b(s)/s^{\gamma_i}$ unbounded near zero. Choose a truncated interval $[a_i, 1/8]$ whose finite supremum M_i exceeds $2i$, and select s_i with quotient greater than $M_i/2$. This gives (13); since $b \le 1$, $s_i^{\gamma_i} < 1/i$ and $s_i \to 0$. Choose an actual $\beta > b(s_i)/2$ and then an actual bad parameter $\lambda_i > \beta/2$. This proves (14) using only approximate suprema. Finally

$$\frac{s_i}{\lambda_i^k} < (4/i)^k s_i^{1-k\gamma_i} \longrightarrow 0.$$

□

Suppose now that such a bad sequence exists. If its p_i have a bounded subsequence, Lemma 2.1 gives a uniform Taylor error $Cr^{1+\theta}$. The nonzero derivative $(Du, 1)$ then makes every $\lambda > Cr^\theta$ good. Thus $\lambda_i \le C' s_i^\theta$, contradicting $\lambda_i > i s_i^{\gamma_i}/4$. Hence we may assume $p_i \to \infty$ and $\epsilon_i \to 0$.

For every fixed multiple of r_i and every center within a fixed multiple of r_i of z_i, the parameter $\Lambda\lambda_i$, $\Lambda = 32$, is eventually good. Indeed for each fixed large M, the boundary distances are comparable and

$$\beta \le b(2Ms_i) \le 8\lambda_i(2M)^{\gamma_i} < 32\lambda_i.$$

Take a base fit P_i, rotate it to $c_i e_N$, and define

$$U_i(z_i + r_i(\sqrt{\lambda_i}x, t)) = U_i(z_i) + r_i c_i[t + \lambda_i V_i(x, t)]. \tag{16}$$

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The domains exhaust $\mathbb{R}^n \times \mathbb{R}$ by (15). The following geometric conclusions of [1, Section 3] apply with (12): the V_i have a locally uniformly convergent subsequence with nonaffine limit V, and V obeys the all-center, all-radius fit condition

$$|V(x+h,t+\ell) - V(x,t) - p\cdot h - q\ell| \le C_0 R \quad (|\ell| \le R,\ |h-p\ell| \le R), \qquad C_0 = 8\Lambda. \tag{17}$$

Here p, q may depend on the center and R; writing $q = a - |p|^2/2$ gives exactly the convention in Theorem A.1.

We spell out the reused compactness interfaces. Equal-parameter concentric fits at comparable radii have direction and vector differences $O(\sqrt{\lambda})$ and norm difference $O(\lambda)$: evaluate their affine errors on a common ball, then on a short common axial segment. The displaced-center version holds in quarter margins. Chaining from r_i to Rr_i costs a factor $1+\log R$. Choose a fixed enclosing R larger than the resulting multiple of $1+\log R$ for each normalized compact box. In the base coordinates, fit coefficients at its centers have $P_x = O(\sqrt{\lambda_i})$ and $P_t - c_i = O(\lambda_i)$. Thus the normalized coefficients $p_i' = P_x/(c_i\sqrt{\lambda_i})$ and $q_i' = (P_t/c_i - 1)/\lambda_i$ are bounded. Their fitted domains contain

$$|\ell| \le S/2, \qquad |h - p_i'\ell/(1+\lambda_i q_i')| \le S,$$

with value error at most $2\Lambda S$. First choose S small and then increments small relative to the bounded coefficients; this proves equicontinuity without derivative bounds on V_i. The enclosing fit gives local boundedness. At each fixed center and radius, take $S = 4R$, extract the bounded fit coefficients, and use strict domain margins to obtain (17). This auxiliary subsequence may depend on that center and radius, since V is already fixed. Finally an affine limit $V = p\cdot x + qt$ would give physical nonzero slopes $c_i(\sqrt{\lambda_i}p, 1+\lambda_i q)$ with residual $o(\lambda_i r_i)$ on their λ_i cylinders, contradicting badness. Only the geometric part of the source's extraction is used; its PDE identification is replaced next.

5 The constrained equation and finite effective viscosity

In the rotated frame put $\xi_i = (a_i, b_i)$, $|a_i|^2 + b_i^2 = 1$, $\beta_i = 1/c_i$, $A_i = 1-\beta_i^2$, $\Pi_i = I - \xi_i\xi_i^T$ and $S_\lambda = \operatorname{diag}(I, \sqrt{\lambda})$. At every test ϕ of V_i, the physical gradient is $c_i q_i$, where

$$q_i = (\sqrt{\lambda_i} D_x\phi, 1+\lambda_i\phi_t), \qquad q_i\cdot\xi_i = \beta_i.$$

The last equality follows from the exact affine increment and holds for both upper and lower tests. Substituting into (10) and dividing by $\lambda_i c_i^3/r_i$ gives

$$D^2\phi[(D_x\phi, 1+\lambda_i\phi_t)] + C_i(D\phi)\operatorname{tr}(S_{\lambda_i}\Pi_i S_{\lambda_i} D^2\phi) = 0, \tag{18}$$
$$C_i(q) = \nu_i + \epsilon_i(2q_t + |q_x|^2) + \epsilon_i\lambda_i q_t^2, \qquad \nu_i = \frac{\epsilon_i A_i}{\lambda_i}.$$

The constraint was used in

$$|\Pi_i q_i|^2 = A_i + \lambda_i(2\phi_t + |D_x\phi|^2) + \lambda_i^2\phi_t^2.$$

This identity is only asserted at actual tests. There, $C_i = \epsilon_i|\Pi_i q_i|^2/\lambda_i \ge 0$. Replacing C_i globally by $\max\{C_i, 0\}$ gives a continuous degenerate-elliptic operator agreeing at every test; this legitimizes off-constraint viscosity stability.

One bounded contact controls the possibly negative parameter A_i. If $|V_i| \le M$ on a fixed unit ball, then $V_i - K|z|^2$, $K > 2M+1$, attains its maximum in the interior. Its quadratic upper test has

uniformly bounded gradient. The preceding nonnegative identity at that point yields $A_i \geq -C_M \lambda_i$, and hence

$$\nu_i \geq -C_M \epsilon_i \longrightarrow 0 \quad \text{from below.} \tag{19}$$

No bounded test gradient is asserted at arbitrary points.

The exact increment also yields

$$V_i(z + sw_i) = V_i(z) + s\delta_i, \quad w_i = (a_i, b_i\sqrt{\lambda_i}), \quad \delta_i = \frac{\beta_i - b_i}{\sqrt{\lambda_i}}.$$

Evaluating at $s = 1/|w_i|$ in a larger fixed ball gives $|\delta_i| \leq C|w_i|$. Pass to subsequences $a_i \to a$, $b_i \to b$, $\delta_i \to \delta$ to obtain

$$V(x + sa, t) = V(x, t) + s\delta. \tag{20}$$

If $a = 0$, then $\delta = 0$.

For a bounded subsequence of ν_i, (19) gives $\nu_i \to \nu \geq 0$. The extended operators converge on compact jet sets, so their limit solves

$$D^2V[(D_xV, 1)] + \nu \operatorname{tr}((I - aa^T)D^2_{xx}V) = 0.$$

If $a = 0$, this is $G_\nu = 0$. The fit condition (17), the pinned Liouville theorem for $\nu = 0$, and Theorem A.1 for $\nu > 0$ make V affine.

If $a \neq 0$, let $e = a/|a|$, $k = \delta/|a|$. The transport identity gives $V(se + y, t) = ks + W(y, t)$, $y \perp e$. Tests $ks + \psi(y, t)$ are genuine tests for V, with Hessian annihilating e. Thus

$$D^2W[(D_yW, 1)] + \nu\Delta_yW = 0.$$

The fits project with exactly the same constant: write an original fit $p = de + p_\perp$ with ordinary time coefficient q and choose $h = d\ell e + h_\perp$. For $|\ell| \leq R$ and $|h_\perp - p_\perp \ell| \leq R$ these increments lie in its box. Subtracting the exact $kd\ell$ term gives

$$|W(y + h_\perp, t + \ell) - W(y, t) - p_\perp \cdot h_\perp - (q + d^2 - kd)\ell| \leq C_0 R.$$

The same rigidity applies in the $n - 1 \geq 1$ transverse dimensions. This again makes V affine, contradicting its construction. All finite-viscosity alternatives, including near-critical original slopes, are excluded.

6 A uniform small-solution estimate for the lift

Lemma 6.1 (Uniform small-solution interface)**.** *Fix N, $\lambda_0 > 0$, $\Lambda_0 < \infty$, $\rho > 0$ and $K < \infty$. Consider autonomous continuous degenerate-elliptic operators $F(X, q)$ with $F(0, q) = 0$. Suppose they are C^2 on the common jet ball $|X|, |q| < \rho$, with common C^2 bound K, and their matrix derivatives have eigenvalues in $[\lambda_0, \Lambda_0]$ there. There are common $\sigma > 0$, $\alpha > 0$ and $C < \infty$ such that any continuous viscosity solution of $F(D^2Z, DZ) = 0$ on B_1 with $\|Z\|_{L^\infty} \leq \sigma$ is $C^{2,\alpha}$ on $B_{1/2}$ with norm at most C. The threshold can be reduced to obtain any fixed sufficiently small bound on the first and second derivatives.*

Proof. This is the common-data form of [4, Theorem 1.3 and Proposition 4.1, pp. 571–573]. The proposition's contradiction argument explicitly allows varying F_k under a common second-derivative bound, extracts F_k, DF_k, and obtains a uniformly elliptic constant-coefficient limiting equation. Its quadratic approximation and iteration constants depend only on the common local data. The lower-order and space variables in that source are absent here. Its H2 uses the matrix operator norm; the eigenvalue assumptions above give its constants λ_0 and $N\Lambda_0$. Reducing the common jet radius gives the last assertion. No gradient bound for the unknown solution is assumed. □

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Lemma 6.2 (Coarse-cylinder estimate)**.** *Fix $n, L, b_0 > 0$. There are $d_*, k_* > 0$ and $C < \infty$, depending only on these data, with the following property. Suppose an actual affine lift U of a weak p-harmonic solution, $p \geq 3$, is defined on a neighborhood of an η-fitted cylinder $Z_\eta(z, H; P)$, with slope $P = ce_N$ after rotation, and*

$$\begin{gathered} c \in [1/2, L+1], \quad A = 1 - c^{-2} > 0, \quad \epsilon = (p-2)^{-1} \leq 1, \quad e = \epsilon A \leq 1, \\ \xi = (a, b), \quad |a|^2 + b^2 = 1, \quad b \geq b_0, \qquad 0 < \eta \leq d_* e. \end{gathered} \tag{21}$$

Then $P_{\mathrm{new}} = DU(z)$ exists and is nonzero, and, whenever $0 < \lambda \leq e$, $0 < r \leq k_* \ell$ with $\ell = (H/4)\sqrt{\eta/e}$, one has

$$\sup_{h:\, z+h \in Z_\lambda(z, r; P_{\mathrm{new}})} |U(z+h) - U(z) - P_{\mathrm{new}} \cdot h| \leq Cer^2/H. \tag{22}$$

Proof. Put $T_e = \mathrm{diag}(\sqrt{e} I, 1)$ and set

$$U(z + \ell T_e(x,t)) = U(z) + c\ell[t + eW(x,t)].$$

For $d_* \leq 1$ this ellipsoid lies strictly in the fitted cylinder: its transverse radius is $H\sqrt{\eta}/4$ and its axial radius is at most $H/4$. The value fit gives

$$\|W\|_{L^\infty(B_1)} \leq (4/c)\sqrt{\eta/e}. \tag{23}$$

At constrained tests its equation is

$$D^2 W[(D_x W, 1 + eW_t)] + B(DW)\,\mathrm{tr}(S_e \Pi S_e D^2 W) = 0,$$

where $S_e = \mathrm{diag}(I, \sqrt{e})$ and

$$B(q) = 1 + \epsilon(2q_t + |q_x|^2) + \epsilon^2 A q_t^2.$$

At those tests $B = |\Pi(\sqrt{e} q_x, 1 + eq_t)|^2/A \geq 0$. Again replace B globally by its nonnegative part. To obtain a proportional, rather than absolute, derivative bound include an amplitude parameter $d \in [0,1]$:

$$\begin{aligned} F_d(X, q) &= X[(dq_x, 1 + edq_t)] + \max\{B_d(q), 0\}\,\mathrm{tr}(S_e \Pi S_e X), \\ B_d(q) &= 1 + \epsilon(2dq_t + d^2|q_x|^2) + \epsilon^2 A d^2 q_t^2. \end{aligned}$$

Allowing ϵ, A, e, d independently in $[0,1]$ only enlarges the family. All members are continuous and degenerate elliptic on all jets. On $|q| \leq 1/8$ one has $B_d \geq 3/4$, so the max switch is absent. They vanish at $X = 0$ and have common local C^2 bounds. At $q = 0$ their matrix form is

$$Q(h,s) = |h|^2 + (1+e)s^2 - (a \cdot h + b\sqrt{e} s)^2, \qquad \frac{b_0^2}{3}(|h|^2 + s^2) \leq Q(h,s) \leq 2(|h|^2 + s^2).$$

Here is a dimension-free certificate for this actual vector form. Set $t = \sqrt{e}$ and $g = |a|^2|h|^2 - \langle a, h\rangle^2 \geq 0$, by Cauchy–Schwarz. The unit relation $|a|^2 + b^2 = 1$ gives

$$Q = |bh - tsa|^2 + s^2 + g$$

and

$$\begin{aligned} 3Q - b^2(|h|^2 + s^2) = |bh - 2tsa|^2 + |bh - tsa|^2 + 3g \\ + (3 - b^2 - 2t^2|a|^2)s^2 \geq 0. \end{aligned}$$

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Indeed $3 - b^2 - 2t^2|a|^2 \geq 1 + b^2$ for $t^2 \leq 1$. The upper bound follows by dropping the negative square in Q. This works on any real inner product space and includes $a = 0$. The supporting Lean theorem `physical_uniform_bounds` certifies precisely these bounds with $t = \sqrt{e}$, the unit relation, $0 \leq e \leq 1$ and $b_0^2 \leq b^2$.

We quantify the common neighborhood as well. Write

$$v(q) = (dq_x, 1 + edq_t), \qquad M_d(q) = v(q)v(q)^T + B_d(q)S_e\Pi S_e.$$

For $|q| \leq r \leq 1/8$, one has $|v(q) - e_N| \leq r$, $|B_d(q) - 1| \leq 2r + 2r^2$, and $\|S_e\Pi S_e\| \leq 1$. Hence

$$\|M_d(q) - M_d(0)\| \leq 4r + 3r^2 < 5r.$$

Thus the explicit common radius

$$\rho = \min\{1/8, b_0^2/30\}$$

gives $B_d \geq 3/4$ and eigenvalues in $[b_0^2/6, 3]$ throughout $|q| \leq \rho$, including all parameter endpoints. For every positive semidefinite matrix Y, the operator-norm convention gives

$$\frac{b_0^2}{6}\|Y\| \leq \operatorname{tr}(M_d(q)Y) \leq 3\operatorname{tr} Y \leq 3N\|Y\|.$$

Savin's ellipticity constants can therefore be taken as $\lambda = b_0^2/6$ and $\Lambda = 3N$. On $|X|, |q| \leq \rho$ the max switch is absent, so the coefficients are polynomial in q; D_qM_d and $D_q^2M_d$ are uniformly bounded. Consequently $D^2_{Xq}F_d = D_qM_d$ and $D^2_{qq}F_d = \operatorname{tr}(D_q^2M_dX)$ have common bounds depending only on N, b_0. Together with $F_d(0, q) = 0$, these verify every hypothesis of Lemma 6.1 with one common K. The square root in S_e causes no loss: parameter derivatives are not required.

Let $a_0 = \|W\| \leq \sigma$, with the common threshold from Lemma 6.1. If $a_0 = 0$ the estimate is immediate. Otherwise take $d = a_0/\sigma$ and $Z = W/d$. Dividing the equation by d gives exactly $F_d(D^2Z, DZ) = 0$, and $\|Z\| = \sigma$. Thus

$$\|DW\|_{B_{1/2}} + \|D^2W\|_{B_{1/2}} \leq Ca_0.$$

Choose d_* so that (23) is below the threshold. Scaling back yields

$$\|T_eD^2UT_e\| \leq \frac{ce}{\ell}C\frac{4}{c}\sqrt{\eta/e} \leq C'e/H$$

on the half ellipsoid. By decreasing the threshold its derivative conclusion also gives $c(1+eW_t) \geq c/2$. Hence

$$P_{\text{new}} = c(\sqrt{e}D_xW(0), 1 + eW_t(0)) \neq 0, \qquad |T_e^{-1}N_{P_{\text{new}}}| \leq C.$$

For h in its λ cylinder, $\lambda \leq e$ implies $|T_e^{-1}h| \leq Cr + \sqrt{\lambda/e}r \leq C'r$. Choose k_* so the whole segment lies in the half ellipsoid. Taylor's formula proves (22). □

7 Infinite effective viscosity and the quadratic comparison

It remains to exclude $\nu_i \to +\infty$. Then $A_i > 0$ eventually. Define

$$R_i = \nu_i^2, \quad H_i = R_ir_i, \quad \kappa_i = 64(4R_i)^{\gamma_i}, \quad \eta_i = \kappa_i\lambda_i.$$

Since $A_i, \epsilon_i \leq 1$, $R_i \leq \lambda_i^{-2}$, and (15) gives $R_is_i \to 0$. Centers within $2H_i$ of z_i retain at least half its boundary distance. At all admissible radii up to H_i, (13) gives

$$\beta \leq b(4R_is_i) \leq 8\lambda_i(4R_i)^{\gamma_i} < \eta_i.$$

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Thus η_i is good throughout these comparisons. Also $\eta_i \le C\lambda_i^{1-2\gamma_i} \to 0$ and $\kappa_i/\nu_i \to 0$. A fixed multiple of λ_i is not asserted good on these growing boxes.

Let $Q_{i,0}$ be an η_i-fit at the base radius r_i, and let P_i be the original $\Lambda\lambda_i$-fit. Their common inscribed ball gives $|P_i - Q_{i,0}| \le C\kappa_i\sqrt{\lambda_i}$, hence the same order for their axis angle. An equal-parameter norm comparison cannot be applied to this pair. Instead take a point on the base axis at distance $t_i r_i$ where $t_i = c\kappa_i^{-1/2}$ and $c > 0$ is sufficiently small and fixed. Its transverse displacement from the coarse axis is at most $Ct_i\kappa_i\sqrt{\lambda_i}r_i \le \sqrt{\eta_i}r_i$; the axial margins also hold. Evaluating both fits there gives

$$|(P_i - Q_{i,0}) \cdot N_{P_i}| \le C\kappa_i^{3/2}\lambda_i.$$

Converting axial projection to norm costs the square of the angle, at most $C\kappa_i^2\lambda_i$. Therefore

$$\big||P_i| - |Q_{i,0}|\big| \le C\kappa_i^2\lambda_i. \tag{24}$$

Now chain η_i-fits through dyadic radii from r_i to H_i. These comparisons have equal parameters and common slope bounds, so the source's quadratic norm comparison is applicable. A final coarse fit Q_i satisfies

$$\big||Q_i| - c_i\big| \le C\{\kappa_i^2\lambda_i + \eta_i(1 + \log\nu_i)\}. \tag{25}$$

Put $c_i' = |Q_i|$, $A_i' = 1 - (c_i')^{-2}$. The map $c \mapsto 1 - c^{-2}$ is uniformly Lipschitz on $[1/2, L+1]$, so

$$\frac{|A_i' - A_i|}{A_i} \le C\epsilon_i\frac{\kappa_i^2 + \kappa_i(1+\log\nu_i)}{\nu_i} \longrightarrow 0.$$

Indeed $\kappa_i = 64\,4^{\gamma_i}\nu_i^{2\gamma_i}$, $\gamma_i \to 0$ and $\nu_i \to \infty$; eventually $4\gamma_i < 1/2$. Thus $A_i' > 0$ and $e_i' = \epsilon_i A_i' \sim \nu_i\lambda_i$. This proves the needed comparison before substituting the coarse parameter into the small-solution lemma.

The coarse fit has $\eta_i/e_i' \sim \kappa_i/\nu_i \to 0$. Its added-coordinate component obeys $b_i' \ge (1-\sqrt{\eta_i})/(L+1)$ by (12). Its ellipsoid scale is

$$\frac{\ell_i}{r_i} = \frac{R_i}{4}\sqrt{\eta_i/e_i'} \sim \frac{1}{4}\nu_i^{3/2}\sqrt{\kappa_i} \longrightarrow \infty.$$

Also $\lambda_i \le e_i'$ eventually. Estimate (22) at r_i therefore has error, divided by $\lambda_i r_i$, bounded by

$$\frac{Ce_i'r_i}{\lambda_i H_i} \sim \frac{C}{\nu_i} \longrightarrow 0.$$

Its nonzero slope is the actual $DU_i(z_i)$. It makes λ_i good, contradicting its selection. Every domain used lies in the original solution region by (15). We have excluded all branches of the bad sequence and proved the required power bound.

8 Gradient extraction and a consequence

The preceding argument yields one power bound $b(s) \le Bs^\gamma$ for the full family $p \ge 3$. Decrease γ so $0 < \gamma \le 1$. At centers in $B_{1/2} \subset \mathbb{R}^{n+1}$ choose common $A_0, r_0 > 0$ so $\lambda = A_0 r^\gamma$ is good for $0 < r \le r_0$. On dyadic radii $r_j = 2^{-j}r_0$, the inscribed balls have radii $\rho_j = \sqrt{A_0}r_j^{1+\gamma/2}$ and error $A_0 r_j^{1+\gamma}$. Comparing consecutive affine functions on a common ball gives

$$|P_j(z) - P_{j+1}(z)| \le Cr_j^{\gamma/2}.$$

The slopes converge uniformly. For $\rho_{j+1} < |h| \le \rho_j$, the residual divided by $|h|$ is at most $Cr_j^{\gamma/2} \le C'|h|^\alpha$, where $\alpha = \gamma/(2+\gamma) \in (0, 1/3]$. Thus U is differentiable with uniform Taylor error $C|h|^{1+\alpha}$. Comparing these Taylor formulas at two centers, at their connecting point and at its translate by $|z - z'|e$ for each unit vector e, gives the α-Hölder bound on DU. Larger separations use the Lipschitz bound. Restrict to $b = 0$ and restore the original oscillation to prove Theorem 1.1.

If u_p converges locally uniformly in value to v with bounded local oscillations, the same Taylor estimate passes to v: at each fixed center, take a convergent subsequence of its bounded slopes. The limiting Taylor inequality identifies a unique derivative with that error. The two-center comparison gives the same exponent for v; it need not agree with the independently supplied infinity-harmonic exponent. On nested balls, finite differences give

$$\|Du_p - Dv\| \le C(h^\alpha + \delta/h), \qquad \delta = \|u_p - v\|_{L^\infty}.$$

For small δ, choose $h = \delta^{1/(1+\alpha)}$ to obtain $\|Du_p - Dv\| \le C\delta^{\alpha/(1+\alpha)}$. A rate in p requires independent value-convergence assumptions.

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9 Verification and scope

The proof relies on the pinned mathematical inputs listed in Section 1. It gives no explicit numerical exponent, boundary estimate, endpoint exponent, or result for the inhomogeneous equation beyond the stated large-p corollary. The literature comparison is bounded; no worldwide priority claim is made.

The accompanying Lean 4.34.1 project is pinned to mathlib commit

`d13f23b723b8a846827a245b89c10fc7d3f11612`.

It proves ten supporting theorems: seven vector inner-product and ellipticity theorems and the original three scalar real-algebra lemmas. The vector certificate applies in any real inner product space. Its `physical_uniform_bounds` theorem directly bounds

$$|h|^2 + (1+e)s^2 - (\langle a, h\rangle + b\sqrt{e}\, s)^2$$

under $|a|^2 + b^2 = 1$, $0 \le e \le 1$ and $b_0^2 \le b^2$, without a coordinate split. The compiled theorems have only the standard Lean dependencies `propext`, `Classical.choice` and `Quot.sound`. They do not formalize the local jet-neighborhood argument, Savin's theorem, the PDE, geometric compactness, source theorems, or the main conclusion. Exact arithmetic scripts check selected identities and reject specified mutations; they do not establish analytic or universal claims. The reproducibility receipt separates these coverage levels.

A Finite-viscosity fitted-class rigidity

This appendix supplies the positive-viscosity input used in the main argument. Its proof of record is I236, *analytic-attempt*, Sections 1–6, previously reviewed in I239. The full argument is carried here rather than replaced by that review verdict. It includes the contact-propagation proof; it does not assume regularity of arbitrary entire fitted solutions. Its geometric inputs are the fit compactness and barycentric-envelope lemmas of [1, Sections 4–5], with the equation changes checked below.

Fix an integer $m \ge 1$ and $0 < \tau < \infty$. For $P = (p, q) \in \mathbb{R}^m \times \mathbb{R}$ and a symmetric matrix X, define

$$A_\tau(p) = \begin{pmatrix} \tau I + pp^T & p \\ p^T & 1 \end{pmatrix}, \qquad G_\tau(P, X) = \operatorname{tr}(A_\tau(p)X).$$

The time-gradient component q does not enter G_τ. We use the positive-Hessian convention: upper tests for subsolutions give $G_\tau \geq 0$, and lower tests for supersolutions give $G_\tau \leq 0$.

Theorem A.1 (Finite-viscosity rigidity)**.** *Let $V \in C(\mathbb{R}^m \times \mathbb{R})$ solve $G_\tau = 0$ in the viscosity sense. Suppose a finite $C_0 \geq 0$ has the following property: for every (x,t) and every $R > 0$, some $p \in \mathbb{R}^m$ and $a \in \mathbb{R}$ satisfy*

$$\left| V(x+h, t+\ell) - V(x,t) - p \cdot h - \left(a - \frac{|p|^2}{2} \right) \ell \right| \leq C_0 R \quad (|\ell| \leq R,\ |h - p\ell| \leq R). \tag{26}$$

Then V is affine. No gradient bound or energy assumption is required. No estimate uniform as τ approaches zero or infinity is asserted.

A.1 Time regularity of spatially semiconcave subsolutions

Lemma A.2. *A continuous G_τ-subsolution with locally uniform spatial semiconcavity is locally Lipschitz in all variables.*

Proof. Work on a product cylinder compactly contained in the domain and a smaller cylinder of base points (x_0, t_0). Choose $r > 0$ so that every $\overline{B}_r(x_0)$ stays inside the larger spatial ball. Enlarge the semiconcavity constant to $K \geq 0$. Continuity supplies a common bound M and a time modulus ω on the larger compact cylinder.

At every base point, a supporting hyperplane to $u(\cdot, t_0) - K|\cdot|^2/2$ gives a vector p with

$$u(y, t_0) \leq u(x_0, t_0) + p \cdot (y - x_0) + \frac{K}{2}|y - x_0|^2 \quad (|y - x_0| \leq r).$$

Evaluating at opposite fixed increments bounds $|p|$ by a common L. The same argument at neighboring points bounds spatial Lipschitz constants uniformly in time. Set $K' = K + 1$ and hold this p fixed in

$$h(t) = \max_{|y - x_0| \leq r} \left\{ u(y,t) - p \cdot (y - x_0) - \frac{K'}{2}|y - x_0|^2 \right\}.$$

At t_0 the unique maximizer is x_0, and boundary competitors have a deficit at least $r^2/2$. Choose a common $s > 0$ with $\omega(s) < r^2/8$ and with the relevant time intervals inside the larger cylinder. For $|t - t_0| < s$, boundary competitors are at most $u(x_0, t_0) - 3r^2/8$, whereas the central competitor is at least $u(x_0, t_0) - r^2/8$. Thus every maximizing point is interior. Also h is continuous and $|h| \leq H := M + Lr + K'r^2/2$.

If a scalar upper test ψ touches h at t and y is a maximizer, then

$$\phi(Y, T) = \psi(T) + p \cdot (Y - x_0) + \frac{K'}{2}|Y - x_0|^2$$

is an upper test of u at (y,t), after adjusting the contact constant. Its mixed Hessian is zero. The subsolution inequality gives

$$0 \leq \psi''(t) + K'\big(|p + K'(y - x_0)|^2 + \tau m\big), \qquad \psi''(t) \geq -C, \quad C = K'\big((L + K'r)^2 + \tau m\big).$$

Consequently $h(t) + C(t - t_0)^2/2$ is convex. The one-dimensional viscosity characterization used here follows directly from the chord criterion: failure of convexity yields an interior upper contact by a strictly concave quadratic added to an affine function. The slopes of this bounded convex function are bounded on the middle half interval by its endpoint secants. For example, $8H/s + 4Cs$ bounds the Lipschitz constant of h there. Since $u(x_0, t) \leq h(t)$ with equality at t_0, this bounds $u(x_0, t) - u(x_0, t_0)$ from above. Repeating at base time t, with the same constants, gives the reverse bound. Combining with the spatial bound proves the lemma. □

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A.2 Preserving contact while making the supersolution Lipschitz

Suppose $u \le v$, u is locally Lipschitz, v is a continuous supersolution, and $u(z_0) = v(z_0)$. Work inside a ball with a compact larger margin. If L_0 bounds the Lipschitz constant of u there, choose $L > L_0$ and a small $r > 0$. Define locally

$$v_L(z) = \min_{|h| \le r} \{v(z+h) + L|h|\}.$$

The minimum is attained, and v_L is continuous. Every competitor obeys

$$v(z+h) + L|h| \ge u(z) + (L - L_0)|h|. \tag{27}$$

Thus $u \le v_L \le v$, and the unique minimizer at z_0 is zero. For any minimizing h,

$$(L - L_0)|h| \le v(z) - u(z).$$

Continuity therefore places all minimizers in $B_{r/2}$ on a small neighborhood of z_0.

A lower test ϕ of v_L at z, pulled back by a minimizing fixed translation h, is a lower test of v at $z+h$, with unchanged gradient and Hessian. The autonomous equation implies that v_L remains a supersolution. If h minimizes at z, then $h + z - z'$ is admissible at a sufficiently close z', so

$$v_L(z') \le v(z+h) + L|h + z - z'| \le v_L(z) + L|z - z'|.$$

Reverse the roles of the two points to obtain local L-Lipschitz continuity. Finally, if $v_L(z) = u(z)$, equation (27) forces its minimizing shift to be zero, whence $v(z) = u(z)$. This last implication recovers contact for the original supersolution.

A.3 The exponential flux and weak inequalities

On the full gradient space put

$$H(p,q) = q + \frac{|p|^2}{2}, \qquad \mathcal{A}(p,q) = e^{H(p,q)/\tau}(p,1).$$

Direct differentiation gives

$$D\mathcal{A}(P) = \frac{e^{H(P)/\tau}}{\tau} A_\tau(p), \qquad \operatorname{div} \mathcal{A}(Df) = \frac{e^{H(Df)/\tau}}{\tau} G_\tau(Df, D^2 f) \tag{28}$$

for smooth f. This exponential structure is classical; see [7]. It is not a new operator introduced here. The derivative is symmetric positive definite. On $|P| \le M$, one may take

$$\lambda_0 = \left[\max\{2, (1+2M^2)/\tau\}\right]^{-1}, \quad \Lambda_0 = M^2 + 1 + \tau,$$
$$\lambda_M = e^{-M/\tau}\lambda_0/\tau, \qquad \Lambda_M = e^{(M+M^2/2)/\tau}\Lambda_0/\tau$$

in $\lambda_M I \le D\mathcal{A}(P) \le \Lambda_M I$. Indeed, $A_\tau(p)[(\zeta, s)] = (p \cdot \zeta + s)^2 + \tau|\zeta|^2$; the lower bound follows from $s^2 \le 2(s + p \cdot \zeta)^2 + 2M^2|\zeta|^2$.

Lemma A.3. *For a locally Lipschitz viscosity subsolution f, $\operatorname{div} \mathcal{A}(Df) \ge 0$ distributionally. For a locally Lipschitz supersolution the inequality is reversed.*

Proof. Fix a compact test support in a region where f is M-Lipschitz, with an additional compact margin. For small $\epsilon > 0$ take the full-variable sup convolution

$$f^\epsilon(z) = \sup_y \left(f(y) - \frac{|z-y|^2}{2\epsilon} \right).$$

Optimizers satisfy $|y - z| \le 2M\epsilon$. On the smaller region f^ϵ is M-Lipschitz, converges uniformly to f, and has $D^2 f^\epsilon \ge -I/\epsilon$ distributionally. Pulling an upper test to an attaining fixed translate shows that f^ϵ is an exact subsolution. This uses autonomy, not convexity in the gradient. At its almost every Alexandrov twice-differentiability point,

$$\operatorname{tr}(D\mathcal{A}(Df^\epsilon)D^2 f^\epsilon) \ge 0.$$

To see this, add a positive arbitrarily small quadratic to its second-order Taylor polynomial and use the viscosity inequality.

Mollify f^ϵ on a smaller region, with mollifications f^ϵ_j. Then $|Df^\epsilon_j| \le M$ and $D^2 f^\epsilon_j \ge -I/\epsilon$, giving the common lower bound

$$\operatorname{div} \mathcal{A}(Df^\epsilon_j) \ge -(m+1)\Lambda_M/\epsilon.$$

Both gradients and Hessians converge almost everywhere to those of f^ϵ. For the Hessian statement, the distributional Hessian of a semiconvex function is a matrix measure; its absolutely continuous part is the Alexandrov Hessian, and mollifications of the singular part converge to zero almost everywhere. This is the mollification fact in [5, Remark 3.3], with the sign reversed from its semiconcave formulation. For every nonnegative smooth compactly supported ϕ, integration by parts and one-sided Fatou give

$$\begin{aligned} -\int \mathcal{A}(Df^\epsilon) \cdot D\phi &= \lim_j \int \phi \operatorname{div} \mathcal{A}(Df^\epsilon_j) \\ &\ge \int \phi \operatorname{tr}(D\mathcal{A}(Df^\epsilon)D^2 f^\epsilon) \ge 0. \end{aligned}$$

In particular the displayed nonnegative density is locally integrable. The favorable singular Hessian contribution has not been deleted.

It remains to pass to the nonlinear flux as $\epsilon \downarrow 0$. At a differentiability point z of f, write an optimizer as $y = z + \epsilon b_\epsilon$, where $|b_\epsilon| \le 2M$. Taylor expansion, uniform for bounded b_ϵ, and comparison with $y = z + \epsilon Df(z)$ give

$$\tfrac{1}{2}|b_\epsilon - Df(z)|^2 = o(1).$$

At a differentiability point of f^ϵ, its gradient is b_ϵ. Along a fixed sequence $\epsilon_j \downarrow 0$, discard the null sets for f and all its convolutions. Gradients then converge almost everywhere and are uniformly bounded, so the smooth flux converges in L^1 on the test support. This proves the first assertion. For supersolutions use inf convolution, whose Hessian has the upper bound I/ϵ. Apply Fatou to the negative divergence and compare its minimizing shift with $-\epsilon Df(z)$. This gives the reversed inequality, without claiming that $-f$ satisfies the same equation. □

A.4 Contact propagation

Theorem A.4. *Let $\Omega \subset \mathbb{R}^{m+1}$ be connected and open. Suppose $u \le v$ are continuous, u is a G_τ-subsolution, v is a G_τ-supersolution, and u is locally uniformly semiconcave in space. If they agree at one interior point, they agree everywhere.*

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Proof. First let both functions be locally Lipschitz and put $d = v - u \geq 0$. On a region where their gradients have norm at most M, define

$$B(z) = \int_0^1 D\mathcal{A}(Du(z) + s(Dv(z) - Du(z)))\, ds.$$

This is measurable, symmetric and uniformly elliptic between $\lambda_M I$ and $\Lambda_M I$. Lemma A.3 and the fundamental theorem of calculus imply

$$\mathcal{A}(Dv) - \mathcal{A}(Du) = BDd, \qquad \int BDd \cdot D\phi \geq 0 \quad (\phi \geq 0).$$

At a zero z_0 of d, choose any sufficiently small ball $B_r(z_0)$ compactly inside this region. Lax–Milgram gives $w \in H^1(B_r)$ with $w - d \in H_0^1(B_r)$ and $\int BDw \cdot D\phi = 0$ for $\phi \in H_0^1(B_r)$. The negative-part test gives $w \geq 0$, because the boundary trace is nonnegative. Testing the difference with $(w-d)^+$ gives

$$\int BD(w-d)^+ \cdot D(w-d)^+ \leq 0,$$

so $0 \leq w \leq d$. Classical interior continuity and Harnack for linear divergence-form weak solutions with bounded measurable symmetric uniformly elliptic coefficients apply [6, Sections 2–3]. For $m = 1$ the total dimension is two. If using a version stated only in dimensions above two, extend w constantly in a third coordinate and extend the coefficient matrix to $\operatorname{diag}(B, 1)$ on a product neighborhood. Fubini verifies the weak equation, with common positive ellipticity; the three-dimensional Harnack inequality then restricts to the original plane. Thus $w(z_0) = 0$ and Harnack, followed by a chain of balls, gives $w = 0$ throughout B_r. Since $w - d \in H_0^1(B_r)$, d has zero trace there. Its Lipschitz continuity on a neighborhood of the closure identifies that trace with its pointwise boundary restriction. Hence $d = 0$ on the entire sphere of radius r. The radius was arbitrary below a fixed margin, so d vanishes on a neighborhood of z_0.

For the stated pair, Lemma A.2 first makes u locally Lipschitz. At every contact point apply the preceding cone construction to v. The Lipschitz-pair argument makes $v_L = u$ nearby, and (27) recovers $v = u$ there. The contact set is consequently open. It is closed by continuity, nonempty by hypothesis, and all of Ω by connectedness. □

A.5 Geometric normalization and the affine-slice alternative

We now prove Theorem A.1. If $C_0 = 0$, fitted open boxes are exactly affine and overlap to give one affine function. Suppose $C_0 > 0$; it may be enlarged to at least one for estimates. For a fitted frame (x_0, t_0, R, p_0, a_0) the transformation

$$U(y,s) = \frac{V(x_0 + R(y + p_0 s), t_0 + Rs) - V(x_0, t_0)}{R} - p_0 \cdot y - \left(a_0 + \frac{|p_0|^2}{2}\right) s \tag{29}$$

preserves the fit constant, changing fit parameters to $(p - p_0, a - a_0)$. Differentiation on test functions gives $G_\tau[U] = RG_\tau[V]$, with τ unchanged. The equation-free fit comparisons and compactness in [1, Section 4] therefore apply: normalized functions have centered growth at most $CK(1 + \log K)^2$ on boxes of radius K, and form a locally compact family. Locally uniform limits of these normalized frames retain both $G_\tau = 0$ and the all-center, all-radius fit (26) with the same C_0. The equation passes by viscosity stability on finite jets. For any fixed center and radius, fit compactness gives a bounded sequence of fit coefficients; extract a convergent subsequence and use strict fitted-domain margins to pass the fit inequality, then continuity to include the boundary. The auxiliary

subsequence may depend on that center and radius, since the limiting function is already fixed. This proves the required fit for every center and radius without choosing one subsequence for uncountably many fits. The growth estimate supplies uniform spatial subquadratic tails on compact time intervals.

If every horizontal slice is affine, write $V(x,t) = P(t)\cdot x + c(t)$, with P, c continuous. A fit at (x,t) of radius R gives $|p - P(t)| \le C_0$ by taking $\ell = 0$. For fixed $\ell > 0$ and $R \ge \ell$, evaluate the same fit at $(h,\ell) = (p\ell, \ell)$ and $(-p\ell, -\ell)$ and add to obtain

$$\begin{aligned}&\big|[P(t+\ell) + P(t-\ell) - 2P(t)]\cdot x\\ &\quad + \ell p\cdot[P(t+\ell) - P(t-\ell)] + c(t+\ell) + c(t-\ell) - 2c(t)\big| \le 2C_0R.\end{aligned}$$

All terms except the coefficient of x are bounded uniformly as x varies. Thus that coefficient vanishes, and continuity makes $P(t) = bt + d$. The centered growth bound, applied at $x = \pm Ke$ and $|t| \le K$, yields $|P(t)| \le C(1 + \log K)^2$ and forces $b = 0$. Tests $P\cdot x + \psi(t)$ then reduce the equation for c to $\psi'' = 0$, so c and V are affine.

Otherwise a horizontal slice has a nonzero midpoint defect. The transformation $V(x,t) \mapsto -V(x,-t)$ preserves the equation and the fits, changes (p,a) to $(-p,a)$, and reverses the defect. Choose its sign positive.

A.6 Extremal envelope and the changed equation

Let $\kappa > 0$ be the supremum, over all times and finite positive-weight probability configurations, of

$$\frac{\sum_j \theta_j V(y_j,t) - V(\sum_j \theta_j y_j, t)}{r}, \qquad r^2 = \sum_j \theta_j \left| y_j - \sum_k \theta_k y_k \right|^2 > 0.$$

It is finite: normalize a radius-r fit at the barycenter using (29). The centered growth bound is at most $C(1+|y|^2)$; integration against a mean-zero, unit-second-moment configuration bounds its normalized defect. Both the quotient and τ are invariant under the fitted-frame transformation.

Take configurations tending to κ, normalized to mean zero and second moment one, and locally uniformly convergent normalized solutions $V_j \to \widehat{V}$. The corresponding measures have a weak limit μ with mean zero and second moment at most one. First-moment uniform integrability preserves the mean; uniform subquadratic tails give uniform integrability for the varying value integrals. Hence

$$\int \widehat{V}(y,0)\, d\mu(y) - \widehat{V}(0,0) = \kappa.$$

For $s > 0$ use the barycentric envelope

$$\mathcal{C}_s V(x,t) = \sup_{\substack{\theta_j > 0,\ \sum \theta_j = 1\\ \sum \theta_j y_j = x}} \left\{ \sum_j \theta_j V(y_j,t) - \frac{1}{2s} \sum_j \theta_j |y_j - x|^2 \right\}.$$

The Jensen bound, also valid in the normalized closure, gives

$$0 \le \mathcal{C}_s V - V \le \sup_{r \ge 0} \left(\kappa r - \frac{r^2}{2s} \right) = \frac{\kappa^2 s}{2}. \tag{30}$$

The measure μ is admissible for the envelope by its geometric measure representation [1, Section 5]. At $s = 1/\kappa$ its defect gives the reverse bound $\kappa/2$. Equality also forces the second moment

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of μ to be one. Finite attainment, continuity and the concave-envelope identity are the geometric conclusions of that same lemma. A radius-$1/\kappa$ frame and time translation put the equality at parameter one and time one. Rename the resulting solution V, and set on $t > 0$

$$W(x,t) = \mathcal{C}_t V(x,t) - \frac{\kappa^2 t}{2}.$$

Then

$$V - \frac{\kappa^2 t}{2} \le W \le V, \qquad W(0,1) = V(0,1) = 0, \qquad \mathcal{C}_1 V(0,1) - V(0,1) = \frac{\kappa^2}{2} > 0. \tag{31}$$

The geometric identity

$$\mathcal{C}_t V(x,t) = \frac{|x|^2}{2t} + \mathrm{Conc}\left[V(\cdot,t) - \frac{|\cdot|^2}{2t}\right](x)$$

gives $D^2_{xx} W \le I/t$. Here Conc denotes the least concave majorant. On each compact positive-time cylinder $t \ge t_{\min} > 0$, this gives spatial semiconcavity with the single constant $1/t_{\min}$, uniformly in space and time. Thus, once the subsolution property below is proved, the hypotheses of Lemma A.2 and the contact theorem apply locally.

We must check the subsolution property for the changed PDE, rather than import the zero-viscosity one. At an upper test ϕ of W, choose a finite attaining configuration with positive weights and barycenter x. The test for its weighted sum is

$$\Phi(Y,t) = \phi\left(\sum_j \theta_j y_j, t\right) + \frac{\kappa^2 t}{2} + \frac{1}{2t}\sum_j \theta_j \left| y_j - \sum_k \theta_k y_k \right|^2.$$

Use the synchronized-time theorem of sums from [1, Sections 2 and 5]. Divide each weighted closed jet by its positive weight, apply its individual subsolution inequality, and multiply the result back by that weight. Test the block matrix inequality on the common-time vector $((q_j)_j, 1)$, with

$$q_j = D_x\phi + \frac{y_j - x}{t},$$

and on each of the m common spatial-translation vectors $((e_k)_j, 0)$. First let the theorem-of-sums error tend to zero, then let the time synchronization parameter tend to infinity. The synchronization Hessian vanishes on all these vectors; large individual time-gradient components do not enter G_τ. The resulting inequality is

$$D^2\Phi[((q_j)_j, 1)] + \tau \sum_{k=1}^{m} D^2\Phi[((e_k)_j, 0)] \ge 0.$$

Along the first vector, the centered offsets scale by $1 + h/t$, so their variance divided by $2(t+h)$ is affine in h. Along common spatial translations the variance is constant. Their weighted velocity is $D_x\phi$. Thus the displayed inequality is exactly $G_\tau(D\phi, D^2\phi) \ge 0$. It proves that W is a subsolution, including at zero horizontal gradient.

Theorem A.4 now applies to (31) on the connected positive-time half-space and gives $W = V$ there. For any fixed $t > 0$, subquadratic spatial growth implies that $V(y,t) - |y|^2/(2t)$ attains a global maximum at some y_*. Average its maximum inequality over any finite configuration with mean y_*. The variance identity yields

$$\sum_j \theta_j V(y_j,t) - \frac{1}{2t}\sum_j \theta_j |y_j - y_*|^2 \le V(y_*,t).$$

The singleton configuration attains equality, so $\mathcal{C}_t V(y_*,t) = V(y_*,t)$. But $W = V$ requires the same gap to be $\kappa^2 t/2 > 0$ at every point, a contradiction. This proves Theorem A.1.

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